# Complément 1: Formulaire

## 1.1 Identités trigonométriques

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta
\end{aligned}$$

$$\begin{aligned}
\sin\alpha\cdot\sin\beta &= -\tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta)\\
\cos\alpha\cdot\cos\beta &= \tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta)\\
\sin\alpha\cdot\cos\beta &= \tfrac{1}{2}\sin(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha-\beta)\\
\cos\alpha\cdot\sin\beta &= \tfrac{1}{2}\sin(\alpha+\beta)-\tfrac{1}{2}\sin(\alpha-\beta)
\end{aligned}$$

$$\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\
\sin\alpha-\sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\\
\cos\alpha+\cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\
\cos\alpha-\cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

## 1.2 Intégration par parties

$$\int udv = uv| - \int vdu$$

## 1.3 Intégration des fonctions trigonométriques et exponentielles

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0 \quad \forall\ n \text{ et } m \text{ entiers}$$

$$\int x\sin x\,dx = \sin x - x\cos x$$

$$\int \sin^2 ax\,dx = -\frac{1}{2a}\cos ax\sin ax + \frac{1}{2}x$$

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x-\cos x)$$

## 1.4 Notions d'analyse vectorielle utiles en physique

Soit $(x, y, z)$ un repère **orthonormé**.

Soit $\vec{a} = a_x(x,y,z)\,\vec{e}_x + a_y(x,y,z)\,\vec{e}_y + a_z(x,y,z)\,\vec{e}_z$ un champ vectoriel.

Soit $f(x,y,z)$ un champ scalaire.

### Définitions

$$\mathbf{grad}\, f = \vec{\nabla} f = \frac{\partial f}{\partial x}\,\vec{e}_x + \frac{\partial f}{\partial y}\,\vec{e}_y + \frac{\partial f}{\partial z}\,\vec{e}_z$$

$$\mathrm{div}\,\vec{a} = \vec{\nabla}\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\mathbf{rot}\,\vec{a} = \vec{\nabla}\wedge\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{(Laplacien scalaire)}$$

$$\nabla^2\vec{a} = \nabla^2 a_x\,\vec{e}_x + \nabla^2 a_y\,\vec{e}_y + \nabla^2 a_z\,\vec{e}_z \quad \text{(Laplacien vecteur)}$$

### Identités vectorielles

$$\mathrm{div}\,\mathbf{grad}\, f = \nabla^2 f$$

$$\mathbf{grad}\,\mathrm{div}\,\vec{a} = \mathbf{rot}\,\mathbf{rot}\,\vec{a} + \nabla^2\vec{a}$$

$$\mathrm{div}\,(f\,\vec{a}) = \vec{a}\cdot\mathbf{grad}\, f + f\,\mathrm{div}\,\vec{a}$$

$$\mathrm{div}\,\mathbf{rot}\,\vec{a} = 0$$

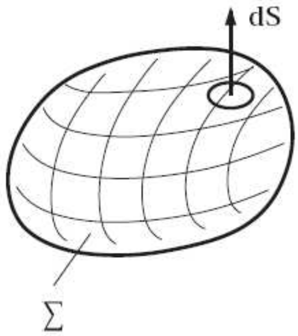
$$\mathbf{rot}\,\mathbf{grad}\, f = 0$$

$$\mathbf{grad}\,(\vec{a}\cdot\vec{a}/2) = (\vec{a}\cdot\vec{\nabla})\vec{a} + \vec{a}\wedge\mathbf{rot}\,\vec{a}$$

$$\mathbf{rot}\,(\vec{a}\wedge\vec{b}) = \vec{a}\,\mathrm{div}\,\vec{b} - \vec{b}\,\mathrm{div}\,\vec{a} + \textstyle\sum_i \dfrac{\partial\vec{a}}{\partial x_i} b_i - \textstyle\sum_i \dfrac{\partial\vec{b}}{\partial x_i} a_i$$

$$\mathbf{rot}\,(f\,\vec{a}) = \mathbf{grad}\, f\wedge\vec{a} + f\,\mathbf{rot}\,\vec{a}$$

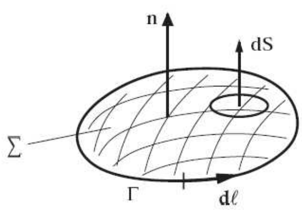
**Théorème de la divergence**




$$\begin{array}{rcl} \Sigma & = & \text{surface fermée} \\ \Omega & = & \text{volume délimité par } \Sigma \\ d\vec{S} & = & \text{élément de surface dirigé} \\ & & \text{vers l'extérieur de } \Omega \\ d\tau & = & \text{élément de volume} \end{array}$$

$$\iint\limits_{\Sigma\,\text{fermé}} \vec{a}\cdot d\vec{S} = \iiint\limits_{\Omega} \operatorname{div}\vec{a}\,d\tau$$

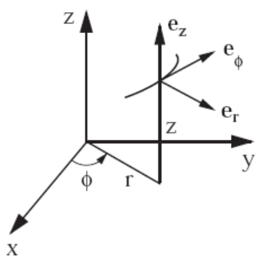
**Formule de Stokes**




$$\begin{array}{rcl} \Gamma & = & \text{contour fermé} \\ \Sigma & = & \text{surface s'appuyant sur } \Gamma \\ d\vec{S} & = & \text{élément de surface orienté} \\ & & \text{selon } \mathbf{n} \\ \mathbf{n} & = & \text{normale d'Ampère} \end{array}$$

$$\oint\limits_{\Gamma} \vec{a}\cdot d\vec{\ell} = \iint\limits_{\Sigma\,\text{ouvert}} \operatorname{rot}\vec{a}\cdot d\vec{S}$$

## 1.5 Coordonnées cylindriques




coordonnées: $r,\ \phi,\ z$

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

Soient $f(r,\phi,z)$ et $\vec{a} = a_r(r,\phi,z)\,\vec{e}_r + a_\phi(r,\phi,z)\,\vec{e}_\phi + a_z(r,\phi,z)\,\vec{e}_z$.

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\vec{e}_\phi + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\operatorname{div}\vec{a} = \frac{1}{r}\frac{\partial (r a_r)}{\partial r} + \frac{1}{r}\frac{\partial a_\phi}{\partial \phi} + \frac{\partial a_z}{\partial z}$$

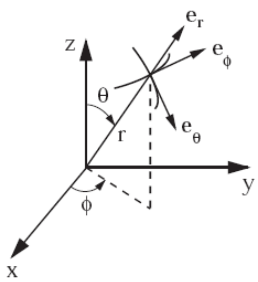
$$\mathrm{rot}\,\vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\phi + \frac{1}{r}\left(\frac{\partial (r a_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\vec{e}_z$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \vec{a} = \left(\nabla^2 a_r - \frac{a_r}{r^2} - \frac{2}{r^2}\frac{\partial a_\phi}{\partial \phi}\right)\vec{e}_r + \left(\nabla^2 a_\phi - \frac{a_\phi}{r^2} + \frac{2}{r^2}\frac{\partial a_r}{\partial \phi}\right)\vec{e}_\phi + (\nabla^2 a_z)\vec{e}_z$$

| | |
|---|---|
| position: | $\vec{r}(r,\phi,z) = r\,\vec{e}_r + z\,\vec{e}_z$ |
| vitesse: | $\vec{v}(r,\phi,z) = \dot{r}\,\vec{e}_r + r\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\vec{e}_z$ |
| accélération: | $\vec{a}(r,\phi,z) = (\ddot{r} - r\dot{\phi}^2)\,\vec{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\vec{e}_\phi + \ddot{z}\,\vec{e}_z$ |

### 1.6 Coordonnées sphériques




coordonnées: $r,\ \theta,\ \phi$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

Soient $f(r,\theta,\phi)$ et $\vec{a} = a_r(r,\theta,\phi)\,\vec{e}_r + a_\phi(r,\theta,\phi)\,\vec{e}_\theta + a_z(r,\theta,\phi)\,\vec{e}_\phi$.

$$\mathrm{grad}\, f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\vec{e}_\phi$$

$$\mathrm{div}\,\vec{a} = \frac{1}{r^2}\frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, a_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi}$$

$$\mathrm{rot}\,\vec{a} = \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta\, a_\phi)}{\partial \theta} - \frac{\partial a_\theta}{\partial \phi}\right)\vec{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \theta} - \frac{1}{r}\frac{\partial (r a_\phi)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_\phi$$

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\mathrm{ctg}\theta\,\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}$$

$$
\begin{aligned}
\nabla^2\vec{a} \quad = \quad & \left[\nabla^2 a_r - \frac{2}{r^2}\left(a_r + \frac{1}{\sin\theta}\frac{\partial(\sin\theta\, a_\theta)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial a_\phi}{\partial\phi}\right)\right]\vec{e}_r \\
& + \left[\nabla^2 a_\theta + \frac{2}{r^2}\left(\frac{\partial a_r}{\partial\theta} - \frac{a_\theta}{2\sin^2\theta} - \frac{\mathrm{ctg}\theta}{\sin\theta}\frac{\partial a_\phi}{\partial\phi}\right)\right]\vec{e}_\theta \\
& + \left[\nabla^2 a_\phi + \frac{2}{r^2\sin\theta}\left(\frac{\partial a_r}{\partial\phi} + \mathrm{ctg}\theta\frac{\partial a_\theta}{\partial\phi} - \frac{a_\phi}{2\,\sin\theta}\right)\right]\vec{e}_\phi
\end{aligned}
$$

$$
\begin{array}{ll}
\text{position:} & \vec{r}(r,\theta,\phi) = r\,\vec{e}_r \\
\text{vitesse:} & \vec{v}(r,\theta,\phi) = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\sin\theta\,\dot{\phi}\,\vec{e}_\phi \\
\text{accélération:} & \vec{a}(r,\theta,\phi) = \quad (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\,\vec{e}_r \\
& \qquad\qquad + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\cos\theta\sin\theta)\,\vec{e}_\theta \\
& \qquad\qquad + (r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta)\,\vec{e}_\phi
\end{array}
$$

## 1.6 Les séries de Fourier

Soit $f$ une fonction périodique de période $T = 2\pi/\omega$ avec au maximum un nombre fini de discontinuités. Elle peut se représenter en série de Fourier:

$$
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t
$$

avec

$$
a_0 = \frac{1}{T}\int_0^T f(t)\,dt, \qquad a_n = \frac{2}{T}\int_0^T f(t)\,\cos n\omega t\,dt, \qquad b_n = \frac{2}{T}\int_0^T f(t)\,\sin n\omega t\,dt
$$

Pour une fonction impaire, $a_n = 0 \quad \forall n$. Pour une fonction paire, $b_n = 0 \quad \forall n$.