Quantum Mechanics I

Week 11 (Solutions)

Spring Semester 2025

1 Eigenstates of the spin along an axis

(a) Consider a spin-1/2 particle. Write the eigenstates of the spin operators $\hat{S}^z$ and $\hat{S}^x$ in the basis of states in which $\hat{S}^z$ is diagonal.

In the basis in which $\hat{S}^z$ is diagonal, the spin operators $\hat{S}^x$, $\hat{S}^y$, $\hat{S}^z$ for spin 1/2 read

$$\hat{S}^\alpha = \frac{\hbar}{2}\hat{\sigma}^\alpha\,, \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{1.1}$$

The eigenstates of $\hat{S}^z$ are

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{1.2}$$

By diagonalizing explicitly the matrix $\hat{\sigma}_x$ we find that the eigenstates of $\hat{\sigma}_x$ are

$$|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \qquad |-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,. \tag{1.3}$$

$|+_x\rangle$ and $|-_x\rangle$ represent physical states in which S^x is precisely known, and is equal, respectively, to $+\hbar/2$ and $-\hbar/2$.

(b) Analyze more generally the eigenstates of the operator $\cos\theta\hat{S}^z + \sin\theta\hat{S}^x$.

More generally, we can find the eigenstates of

$$\cos\theta\hat{S}^z + \sin\theta\hat{S}^x = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{1.4}$$

by diagonalizing explicitly the 2×2 matrix. The eigenvalues satisfy the equation $(\lambda - \hbar/2\cos\theta)(\lambda + \hbar/2\cos\theta) - \hbar^2/4\sin^2\theta = \lambda^2 - \hbar^2/4 = 0$, so $\lambda = \pm\hbar/2$. This is expected because the operator $\cos\theta\hat{S}^x + \sin\theta\hat{S}^x$ is just the projection of the spin along an axis which is tilted relative to the z axis. We know that for a spin-1/2 particle the projection of the spin in *any* direction can take only two values: $\pm\hbar/2$. We could measure the value of $\cos\theta\hat{S}^z + \sin\theta\hat{S}^x$ by passing the particle through a Stern-Gerlach apparatus whose axis is tilted by an angle θ relative to the z axis.

The eigenstates are, for $\lambda = \pm 1$, proportional to

$$\begin{pmatrix} \sin\theta \\ -(\cos\theta - \lambda) \end{pmatrix} \tag{1.5}$$

Using trigonometric identities the expressions for the eigenstates can be rewritten as

$$2\begin{pmatrix}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ \sin^2\frac{\theta}{2}\end{pmatrix}\quad \text{for } \lambda = 1\ , \qquad 2\begin{pmatrix}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ -\cos^2\frac{\theta}{2}\end{pmatrix}\quad \text{for } \lambda = -1\ . \tag{1.6}$$

The normalized eigenstates therefore can be chosen as

$$|+_\theta\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \qquad |-_\theta\rangle = \begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}\ . \tag{1.7}$$

The eigenstates $|\pm_\theta\rangle$ are states in which the spin projection along an axis rotated by an angle θ is perfectly known and is equal to, respectively, $\pm\hbar/2$.

(c) Show that the eigenstates found in part b) can be derived by applying a rotation matrix to the state $|+\rangle$, $|-\rangle$ (the eigenstates of $\hat{S}^z$).

The eigenstates can be obtained also by applying a rotation matrix to $|+\rangle$, $|-\rangle$. We need, in particular, a rotation of angle θ with axis directed along y. The rotation operator (acting on the Hilbert space, which in the case considered here is a 2-dimensional space spanned by $|+\rangle$, $|-\rangle$), is

$$\hat{D}_y(\theta) = e^{-i\theta\hat{S}_y/\hbar} = e^{-i\theta\hat{\sigma}_y/2} = \cos\frac{\theta}{2}\hat{1} - i\sin\frac{\theta}{2}\hat{\sigma}_y\ , \tag{1.8}$$

where $\hat{1}$ stands for the 2×2 identity matrix.

Applying the rotation to $|+\rangle$ and $|-\rangle$ gives

$$\begin{aligned}\hat{D}_y(\theta)|+\rangle &= \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle = |+_\theta\rangle\ ,\\ \hat{D}_y(\theta)|-\rangle &= \cos\frac{\theta}{2}|-\rangle - \sin\frac{\theta}{2}|+\rangle = |-_\theta\rangle\ .\end{aligned} \tag{1.9}$$

in agreement with the eigenstates determined before.

Remarks. Eqs. (1.8) can be interpreted by saying that $\hat{D}_y(\theta)$ performs an "active" rotation on the state vector $|+\rangle$, which has the effect of rotating states oriented in the z direction onto states in the θ direction. However one can equivalently interpret the transformation using a "passive" point of view. In this view, we can say that we are re-expressing the basis of the Hilbert space, using as basis elements $|+_\theta\rangle$ and $|-_\theta\rangle$ instead of $|+\rangle$ and $|-\rangle$. The matrix elements of the transformation matrix are

$$\begin{vmatrix}\langle+|\hat{D}_y(\theta)|+\rangle & \langle+|\hat{D}_y(\theta)|-\rangle\\ \langle-|\hat{D}_y(\theta)|+\rangle & \langle-|\hat{D}_y(\theta)|-\rangle\end{vmatrix} = \begin{vmatrix}\langle+|+_\theta\rangle & \langle+|-_\theta\rangle\\ \langle-|+_\theta\rangle & \langle-|-_\theta\rangle\end{vmatrix}\ . \tag{1.10}$$

These allow to reexpress any state $|\psi\rangle = \sum_{a=\pm}\psi_a|a\rangle = \sum_{a=\pm}\sum_{b=\pm}\psi_a|b_\theta\rangle\langle b_\theta|a\rangle$. Note however that one needs the matrix $\langle b_\theta|a\rangle$, which is the Hermitian conjugate (or equivalently the inverse) of $\hat{D}_y(\theta)$. Any representation is valid provided that it is carried out consistently.

Note also that the components of the angular momentum transform as follows:

$$\begin{aligned}\hat{D}_y^\dagger(\theta)\hat{S}_z\hat{D}_y(\theta) &= \frac{\hbar}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\hat{\sigma}_y\right)\hat{\sigma}_z\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\hat{\sigma}_y\right) \\ &= \cos\theta\hat{S}_z - \sin\theta\hat{S}_x \ ,\end{aligned} \tag{1.11}$$

and similarly

$$\begin{aligned}\hat{D}_y^\dagger(\theta)\hat{S}_y\hat{D}_y(\theta) &= \hat{S}_y \ , \\ \hat{D}_y^\dagger(\theta)\hat{S}_x\hat{D}_y(\theta) &= \sin\theta\hat{S}_z + \cos\theta\hat{S}_x \ .\end{aligned} \tag{1.12}$$

These imply that

$$\hat{D}_y^\dagger(\theta)(\cos\theta\hat{S}_z + \sin\theta\hat{S}_x)\hat{D}_y(\theta) = \hat{S}_z \ . \tag{1.13}$$

2 Spin-1 particles

Consider a particle with spin quantum number $s = 1$. Let $\hat{s}_x$, $\hat{s}_y$, and $\hat{s}_z$ be the matrices of spin $s = 1$ in the representation $(\hat{s}^2, \hat{s}_z)$, where the matrices $\hat{s}^2$ and $\hat{s}_z$ are diagonal.

(a) Find the matrices $\hat{s}_x$, $\hat{s}_y$, and $\hat{s}_z$ in this representation.

We consider the basis $\{|s,m\rangle\}$ where $s = 1$ and $m = +1, 0, -1$, and correspond to the eigenvectors of the operator $\hat{s}_z$. In this basis, $\hat{s}_z$ is diagonal and its components are given by

$$\langle s,m'|\hat{s}_z|s,m\rangle = m\delta_{m',m},$$

where $\delta_{m,m'}$ is the Kronecker delta. To find the expressions of the operators $\hat{s}_x$ and $\hat{s}_y$, we use the relations for the spin angular momentum

$$\begin{aligned}\hat{s}_+|s,m\rangle &= \sqrt{s(s+1) - m(m+1)}|s,m+1\rangle, \\ \hat{s}_-|s,m\rangle &= \sqrt{s(s+1) - m(m-1)}|s,m-1\rangle,\end{aligned}$$

where $\hat{s}_+ = \hat{s}_x + i\hat{s}_y$ and $\hat{s}_- = \hat{s}_x - i\hat{s}_y$. The components of $\hat{s}_+$ and $\hat{s}_-$ are then

$$\begin{aligned}\langle s,m'|\hat{s}_+|s,m\rangle &= \sqrt{s(s+1) - m(m+1)}\delta_{m',m+1} = \sqrt{s(s+1) - mm'}\delta_{m',m+1}, \\ \langle s,m'|\hat{s}_-|s,m\rangle &= \sqrt{s(s+1) - m(m-1)}\delta_{m',m-1} = \sqrt{s(s+1) - mm'}\delta_{m'+1,m}.\end{aligned}$$

which give the matrix representation of the x-component of the spin operator:

$$\begin{aligned}\langle s,m'|\hat{s}_x|s,m\rangle &= \frac{1}{2}\left(\langle s,m'|\hat{s}_+|s,m\rangle + \langle s,m'|\hat{s}_-|s,m\rangle\right) \\ &= \frac{1}{2}\sqrt{s(s+1) - mm'}(\delta_{m',m+1} + \delta_{m'+1,m}),\end{aligned}$$

and the y-component of the spin operator:

$$\begin{aligned}\langle s,m'|\hat{s}_y|s,m\rangle &= \frac{1}{2i}\left(\langle s,m'|\hat{s}_+|s,m\rangle - \langle s,m'|\hat{s}_-|s,m\rangle\right) \\ &= \frac{1}{2i}\sqrt{s(s+1) - mm'}(\delta_{m',m+1} - \delta_{m'+1,m}).\end{aligned}$$

In our case $s = 1$, $m = +1, 0, -1$, so we find

$$\hat{s}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{s}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.1}$$

(b) Show that

$$[\hat{s}_i, \hat{s}_j] = i\epsilon_{ijk}\hat{s}_k, \tag{2.2}$$

where ϵ_{ijk} is the Levi-Civita symbol.

To verify this commutation relation, we explicitly compute the three commutators resulting from the three components of the spin:

$$\hat{s}_x\hat{s}_y = \frac{1}{2} \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}, \quad \hat{s}_y\hat{s}_x = \frac{1}{2} \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} \quad \Rightarrow \quad [\hat{s}_x, \hat{s}_y] = i\hat{s}_z,$$

$$\hat{s}_y\hat{s}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{s}_z\hat{s}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} \quad \Rightarrow \quad [\hat{s}_y, \hat{s}_z] = i\hat{s}_x,$$

$$\hat{s}_z\hat{s}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \hat{s}_x\hat{s}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad [\hat{s}_z, \hat{s}_x] = i\hat{s}_y.$$

and thus we obtain the desired result.

(c) Show that

$$\hat{s}_z^3 = \hat{s}_z \quad \text{and} \quad (\hat{s}_x \pm i\hat{s}_y)^3 = 0. \tag{2.3}$$

What do these equations imply?

We verify these expressions by explicitly computing the matrix-matrix products. For the first one, we have:

$$\hat{s}_z^3 = \hat{s}_z^2\hat{s}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hat{s}_z,$$

From this result, we obtain the eigenvalues of the s_z, since from the eigenvalue equation, we get $m^3 - m = 0$, which yields the expected values of $m = 0, \pm 1$.

For the second expression, we have:

$$(\hat{s}_x + i\hat{s}_y)^3 = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right)^3 = 0,$$

$$(\hat{s}_x - i\hat{s}_y)^3 = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 2 & 0 & 0\\ 0 & 2 & 0\end{pmatrix}\right)^3 = 0.$$

This result is equivalent to $s_\pm^3 = 0$ which shoes that the ladder operators cannot be applied indefinitely, but rather act on a finite ladder of angular momentum states. After at most 2 applications of these operators, you are guaranteed to obtain a result of zero. The result $s_\pm^3 = 0$ will be true on all eigenstates of s_z.

(d) Consider a particle with spin $S = \hbar s$, placed in an external magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$, and the corresponding Hamiltonian operator is $H = g\mathbf{B}\cdot\mathbf{S}$. Ignore all spatial degrees of freedom. Find the time-evolved state of the particle at times $t > 0$, if the particle is initially in the state $|s = 1, m_s = 1\rangle$.

For the parameters given in this problem, the Hamiltonian reduces to:

$$\hat{H} = gB\hat{S}_x.$$

By diagonalizing the matrix S_x, we find the eigenvectors expressed in the S_z basis:

$$|s_x = 1\rangle = \frac{1}{2}\left(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle\right) \tag{2.4}$$

$$|s_x = 0\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right) \tag{2.5}$$

$$|s_x = -1\rangle = \frac{1}{2}\left(|1,1\rangle - \sqrt{2}|1,0\rangle + |1,-1\rangle\right) \tag{2.6}$$

The eigenvalues are simply $0, \pm\hbar$. We write the inverse relations, where we express the S_z states in terms of the S_x ones:

$$|1,1\rangle = \frac{1}{2}\left(|s_x = 1\rangle + |s_x = -1\rangle + \sqrt{2}|s_x = 0\rangle\right) \tag{2.7}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|s_x = 1\rangle - |s_x = -1\rangle\right) \tag{2.8}$$

$$|1,-1\rangle = \frac{1}{2}\left(|s_x = 1\rangle + |s_x = -1\rangle - \sqrt{2}|s_x = 0\rangle\right) \tag{2.9}$$

Applying the time-evolution operator on the initial state (expressed in terms of the S_x eigenstates), we find:

$$|\psi(t)\rangle = e^{-igBtS_x/\hbar}|1,1\rangle = \frac{1}{2}\left(e^{-igBt}|s_x = 1\rangle + e^{igBt}|s_x = -1\rangle + \sqrt{2}|s_x = 0\rangle\right) \tag{2.10}$$

(e) What is the probability of finding the particle in the state $|s = 1, m_s = -1\rangle$?

We transform back to the S_z eigenstates and thus obtain:

$$|\psi(t)\rangle = \cos^2(gBt/2)\,|1,1\rangle - \sin^2(gBt/2)\,|1,-1\rangle - i\sqrt{2}\sin(gBt/2)\cos(gBt/2)\,|1,0\rangle.$$

The probability of finding the particle in the S_z eigenstate $|1,-1\rangle$ is

$$P_{1,-1} = \sin^4(gBt/2). \tag{2.11}$$

3 Infinitesimal Rotation

Consider the angular momentum eigenstate $|j, j_z = j\rangle$, on which we apply a rotation $\hat{D}_y(\epsilon)$ of an infinitesimal angle ϵ around the y-axis. Find the expression up to order ϵ^2 for the probability that the new rotated state is found in the original state:

$$\left| \langle j, j|\hat{D}_y(\epsilon)|j, j\rangle \right|^2 . \tag{3.1}$$

Hint: Use the Taylor expansion of the exponential.
The new state after an infinitesimal rotation about the y-axis is

$$\hat{D}_y(\epsilon)\,|j, j\rangle = \exp\left(-\frac{i\hat{J}_y\,\epsilon}{\hbar}\right)|j, j\rangle. \tag{3.2}$$

Expanding the exponential to second order,

$$\begin{aligned}\hat{D}_y(\epsilon)\,|j, j\rangle &= \Big(1 - \frac{i}{\hbar}\hat{J}_y\epsilon - \frac{\hat{J}_y^2\epsilon^2}{2\hbar^2} + \mathcal{O}(\epsilon^3)\Big)|j, j\rangle \\ &= \Big(1 - \frac{\hat{J}_+ - \hat{J}_-}{2\hbar}\epsilon + \frac{\hat{J}_+^2 + \hat{J}_-^2 - \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+}{8\hbar^2}\,\epsilon^2 + \mathcal{O}(\epsilon^3)\Big)|j, j\rangle,\end{aligned}$$

where we used $\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$. Using $\hat{J}_+|j, j\rangle = 0$ and $\langle j, m'|j, m\rangle = \delta_{m',m}$, we obtain:

$$\begin{aligned}\langle j, j|\hat{D}_y(\epsilon)|j, j\rangle &= \Big\langle j, j\Big|1 - \frac{\epsilon^2}{8\hbar^2}\hat{J}_+\hat{J}_- + \mathcal{O}(\epsilon^3)\Big|j, j\Big\rangle \\ &= 1 - \frac{\epsilon^2}{8\hbar^2}\big(\sqrt{2j}\hbar\big)^2 + \mathcal{O}(\epsilon^3) \\ &= 1 - \frac{\epsilon^2 j}{4} + \mathcal{O}(\epsilon^3).\end{aligned}$$

Therefore

$$\big|\langle j, j|\hat{D}_y(\epsilon)|j, j\rangle\big|^2 = 1 - \frac{\epsilon^2 j}{2} + \mathcal{O}(\epsilon^3). \tag{3.3}$$

Note that the only state that mixes with $\hat{D}_y(\epsilon)|j, j\rangle$ to first order in ϵ is $|j, j-1\rangle$; consequently

$$\big|\langle j, j-1|\hat{D}_y(\epsilon)|j, j\rangle\big|^2 = 1 - \big|\langle j, j|\hat{D}_y(\epsilon)|j, j\rangle\big|^2 + \mathcal{O}(\epsilon^3). \tag{3.4}$$

4 Rotation Operations

Consider an operator V that satisfies the commutation relation

$$[L_i, V_j] = i\hbar\epsilon_{ijk}V_k. \tag{4.1}$$

This is by definition a vector operator (for example $V_j = r_j, p_j, L_j$).

(a) Prove that the operator $e^{-i\phi L_x/\hbar}$ is a rotation operator corresponding to a rotation around the x-axis by an angle ϕ, by showing that

$$e^{-i\phi L_x/\hbar} V_i e^{i\phi L_x/\hbar} = R_{ij}(\phi) V_j \tag{4.2}$$

where $R(\phi)$ is the corresponding rotation matrix. Find that matrix. *Hint: Define $X_i = e^{-i\phi L_x/\hbar} V_i e^{i\phi L_x/\hbar}$, take the derivative with respect to ϕ and solve the resulting differential equation.*

Consider the operator

$$X_i = e^{-i\phi L_x/\hbar} V_i e^{i\phi L_x/\hbar}$$

as a function of ϕ and differentiate it with respect to ϕ. We get

$$\frac{dX_i}{d\phi} = -\frac{i}{\hbar} e^{-i\phi L_x/\hbar} [L_x, V_i] e^{i\phi L_x/\hbar} = \epsilon_{ijk} X_j$$

From this we obtain

$$X_x(\phi) = X_x(0) = V_x$$
$$X_y(\phi) = X_y(0)\cos\phi + X_z(0)\sin\phi = V_y\cos\phi + V_z\sin\phi$$
$$X_z(\phi) = X_z(0)\cos\phi - X_y(0)\sin\phi = V_z\cos\phi - V_y\sin\phi$$

or

$$e^{-i\phi L_x/\hbar} V_i e^{i\phi L_x/\hbar} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = R_{ij} V_j$$

Clearly, the matrix R is a rotation matrix corresponding to a rotation around the x-axis by an angle ϕ.

(b) Using the result of part (a), prove that by applying a rotation of π about the x axis on the state $|l, m\rangle$ yields $|l, -m\rangle$, up to a possible factor of modulus 1, i.e.

$$e^{i\pi L_x/\hbar}|l, m\rangle = e^{i\varphi_m}|l, -m\rangle. \tag{4.3}$$

Setting $\phi = \pi$ in the result of the part (a) and considering the L_z, we get

$$e^{-i\pi L_x/\hbar} L_z e^{i\pi L_x/\hbar} = -L_z. \tag{4.4}$$

Acting on the rotated state with L_z, we get

$$L_z(e^{i\pi L_x/\hbar}|l, m\rangle) = -e^{i\pi L_x/\hbar} L_z|l, m\rangle = -\hbar m(e^{i\pi L_x/\hbar}|l, m\rangle)$$

where we have used Eq. (4.4). From this result, we can identify the rotated state as the eigenstate with $m \to -m$, so

$$e^{i\pi L_x/\hbar}|l, m\rangle = e^{i\varphi_m}|l, -m\rangle \ .$$

Since the matrix $e^{i\pi L_x}$ is unitary, it preserves the normalization of state vectors. So $e^{i\varphi_m}$ must be a complex number of modulus 1 ("a phase"). This implies that φ_m

must be real-valued. However the value of the phases φ_m cannot be determined by the calculation above.

Interpretation. The operator L_z measures angular momentum along the z-axis. When we apply a rotation by π around the x-axis, the coordinate system is reflected such that $z \to -z$. As a result, the rotated observable becomes $-L_z$, meaning the measurement direction has flipped. Consequently, the eigenvalues of L_z must also change sign. This symmetry implies that an eigenstate $|l, m\rangle$ is mapped to $|l, -m\rangle$, reflecting the fact that angular momentum now points in the opposite direction along z. This transformation is a natural consequence of how angular momentum operators behave under spatial rotations.

(c) Show that a rotation by π around the z-axis can also be achieved by first rotating around the x-axis by $\pi/2$, then rotating around the y-axis by π, and, finally, rotating back by $-\pi/2$ around the x-axis. In terms of rotation operators, this is expressed as

$$e^{i\pi L_x/2\hbar} e^{-i\pi L_y/\hbar} e^{-i\pi L_x/2\hbar} = e^{-i\pi L_z/\hbar}.$$

Hint: Use the result from part (a).

We expand the exponential with L_y using the known Taylor series:

$$e^{i\pi L_x/2\hbar} e^{-i\pi L_y/\hbar} e^{-i\pi L_x/2\hbar} = e^{i\pi L_x/2\hbar} \sum_n \frac{(-i\pi)^n}{\hbar^n\, n!} L_y^n e^{-i\pi L_x/2\hbar}. \tag{4.5}$$

We now use the result of part (a) to find the rotation of the operator $V_y = L_y$. By setting $\phi = \pi/2$ in the rotation matrix, we get:

$$e^{i\pi L_x/2\hbar} L_y e^{-i\pi L_x/2\hbar} = L_z. \tag{4.6}$$

This result can be generalized to

$$e^{i\pi L_x/2\hbar} (L_y)^n e^{-i\pi L_x/2\hbar} = (L_z)^n.$$

We use this fact in Eq. (4.5) and find:

$$e^{i\pi L_x/2\hbar} e^{-i\pi L_y/\hbar} e^{-i\pi L_x/2\hbar} = \sum_n \frac{(-i\pi)^n}{\hbar^n\, n!} L_z^n = e^{-i\pi L_z/\hbar}. \tag{4.7}$$

which proves that indeed a rotation by π around the z-axis can also be achieved by first rotating around the x-axis by $\pi/2$, then rotating around the y-axis by π, and, finally, rotating back by $-\pi/2$ around the x-axis

(d) Now consider an electron with total angular momentum $J = L + S$. Let $|\Psi\rangle$ be its state and show that, if we rotate it by π around the z-axis, then by π around the y-axis, and finally, by π around the x-axis, we retrieve the same state with an additional phase factor.

In the case of an electron the rotation operators involve the total angular momentum $J = L + S$. Then, the above statement is expressed mathematically as follows:

$$e^{-i\pi J_x/\hbar} e^{-i\pi J_y/\hbar} e^{-i\pi J_z/\hbar} |\Psi\rangle = e^{-i\pi\sigma_1/2} e^{-i\pi\sigma_2/2} e^{-i\pi\sigma_3/2} e^{-i\pi L_x/\hbar} e^{-i\pi L_y/\hbar} e^{-i\pi L_z/\hbar} |\Psi\rangle.$$

Using the following property, which we have shown in a previous exercise set,

$$e^{i\alpha\sigma_j} = \cos\alpha \mathbb{1} + i\sin\alpha\sigma_j \tag{4.8}$$

where $\alpha = -\pi/2$ (for our case), we obtain:

$$e^{-i\pi J_x/\hbar}e^{-i\pi J_y/\hbar}e^{-i\pi J_z/\hbar}|\Psi\rangle = i\sigma_1\sigma_2\sigma_3 e^{-i\pi L_x/\hbar}e^{-i\pi L_y/\hbar}e^{-i\pi L_z/\hbar}|\Psi\rangle. \tag{4.9}$$

The product $i\sigma_1\sigma_2\sigma_3$ evaluates to $-\mathbb{1}$. Then, we can write:

$$e^{-i\pi J_x/\hbar}e^{-i\pi J_y/\hbar}e^{-i\pi J_z/\hbar}|\Psi\rangle = -e^{-i\pi L_x/2\hbar}\left(e^{-i\pi L_x/2\hbar}e^{-i\pi L_y/\hbar}\right)e^{-i\pi L_z/\hbar}|\Psi\rangle, \tag{4.10}$$

and using the result of part (c) in the parenthesis, we find:

$$e^{-i\pi J_x/\hbar}e^{-i\pi J_y/\hbar}e^{-i\pi J_z/\hbar}|\Psi\rangle = -e^{-i\pi L_x/2\hbar}\left(e^{-i\pi L_z/\hbar}e^{-i\pi L_x/2\hbar}\right)e^{-i\pi L_z/\hbar}|\Psi\rangle. \tag{4.11}$$

Now, the angular momentum takes integer values, thus we may show that $e^{-i2\pi L_z/\hbar} = \mathbb{1}$. We can use this result to write:

$$e^{-i\pi J_x/\hbar}e^{-i\pi J_y/\hbar}e^{-i\pi J_z/\hbar}|\Psi\rangle = -e^{-i\pi L_x/2\hbar}\left(e^{-i\pi L_z/\hbar}e^{-i\pi L_x/2\hbar}e^{i\pi L_z/\hbar}\right)|\Psi\rangle. \tag{4.12}$$

Then, the exponential $e^{-i\pi L_x/2\hbar}$ is appropriately being rotated by an angle π about the z-axis, thus:

$$e^{-i\pi J_x/\hbar}e^{-i\pi J_y/\hbar}e^{-i\pi J_z/\hbar}|\Psi\rangle = -e^{-i\pi L_x/2\hbar}e^{i\pi L_x/2\hbar}|\Psi\rangle = -|\Psi\rangle.$$