General Physics: Electromagnetism, Correction 5

Exercise 1 :

Two particles, with charges 20.0 nC and −20.0 nC, are placed at the points with coordinates (0, 4.00 cm) and (0, -4.00 cm), as shown in Fig.1. A particle with charge 10.0 nC is located at the origin.

1. Find the electric potential energy of the configuration of the three fixed charges.
2. A fourth particle, with a mass of 2.00×10^{-13} kg and a charge of 40.0 nC, is released from the rest at the point (3.00 cm, 0). Find its speed after it has moved freely to a very large distance.

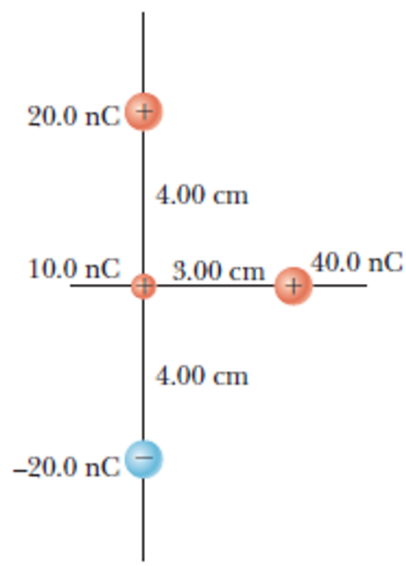

Figure 1: Schematic of the charges and their positions.

Solution 1 :

1. In an empty space, the first charge ($q_1 = +20.0$ nC) is placed at its location with no energy requirement. This charge now creates an electric potential:

$$V_1(r) = k_e \frac{q_1}{r} \tag{1}$$

To place the second charge ($q_2 = +10.0$ nC) at its position, energy needs to be invested:

$$U_{12} = q_2 \cdot V_1(r_{12}) = (10 \cdot 10^{-9} C) \cdot \frac{(8.99 \cdot 10^9\ Nm^2/C^2)(20.0 \cdot 10^{-9} C)}{0.04\ m} = 4.50 \cdot 10^{-5}\ J \tag{2}$$

Finally, to place the third charge ($q_3 = -20$ nC) at its position requires energy:

$$U_{13} + U_{23} = q_3 V_1(r_{13}) + q_3 V_2(r_{23}) = q_3 k_e \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right) = \quad (3)$$

$$= (-20.0 \cdot 10^{-9})(8.99 \cdot 10^9 \; Nm^2/C^2) \left(\frac{20.0 \cdot 10^{-9} C}{0.08 \, m} + \frac{10.0 \cdot 10^{-9} C}{0.04 \, m} \right) = -9.0 \cdot 10^{-5} \; J \quad (4)$$

The total energy of the system of three charges is equal to:

$$U_{tot} = U_{12} + U_{13} + U_{23} = \boxed{-4.5 \cdot 10^{-5} \; J} \quad (5)$$

2. The three fixed charges create a potential where the fourth charge is released:

$$V = V_1 + V_2 + V_3 \quad (6)$$

$$= k_e \frac{q_1}{r_{14}} + k_e \frac{q_2}{r_{24}} + k_e \frac{q_3}{r_{34}} \quad (7)$$

$$= k_e \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right) \quad (8)$$

$$= (8.99 \cdot 10^9 \; Nm^2/C^2) \left(\frac{20.0 \cdot 10^{-9} C}{\sqrt{0.04^2 + 0.03^2} \, m} + \frac{10.0 \cdot 10^{-9} C}{0.03 \, m} - \frac{20.0 \cdot 10^{-9} C}{\sqrt{0.04^2 + 0.03^2} \, m} \right) \quad (9)$$

$$= 3.0 \cdot 10^3 \; V \quad (10)$$

Conservation of energy is used to find the velocity of the fourth charge to find the speed at the great distance from the other three charges:

$$\frac{1}{2} m v_i^2 + q_4 V_i = \frac{1}{2} m v_f^2 + q_4 V_f \quad (11)$$

With $v_i = 0$ and $V_f = 0$:

$$q_4 V_i = \frac{1}{2} m v_f^2 \quad (12)$$

$$v_f = \sqrt{\frac{2 q_4 V_i}{m}} = \sqrt{\frac{2 \cdot 20 \cdot 10^{-9} C \cdot 3.0 \cdot 10^3 \; V}{2 \cdot 10^{-13} \; kg}} = \boxed{3.46 \cdot 10^4 \; m/s} \quad (13)$$

Exercise 2 :

A nonconducting sphere of radius r_0 carries a total charge Q distributed uniformly throughout its volume. Determine the electric potential as a function of the distance r from the center of the sphere for: (a) $r > r_0$ and (b) $r < r_0$. Take $V = 0$ at $r = \infty$; (c) Plot V versus r and E versus r.

- **Hint**: calculate the electric field first, and then the potential from it.

Solution 2 :

(a) The electric field outside a charged, spherically symmetric volume is the same as for a point charge of the same charge magnitude. Integrating the electric field from infinity to the radius of interest will give the potential at that radius.

$$E(r \geq r_0) = \frac{Q}{4\pi\varepsilon_0 r^2} \rightarrow V(r \geq r_0) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r^2} dr = \left.\frac{Q}{4\pi\varepsilon_0 r}\right|_\infty^r = \boxed{\frac{Q}{4\pi\varepsilon_0 r}} \tag{14}$$

(b) Inside the sphere the electric field is obtained from Gauss's law using the charge enclosed by a sphere of radius r.

$$4\pi r^2 E = \frac{Q}{\varepsilon_0} \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi r_0^3} \longrightarrow \boxed{E(r < r_0) = \frac{Qr}{4\pi\varepsilon_0 r_0^3}} \tag{15}$$

Integrating the electric field from the surface (i.e at r_0) to $r < r_0$ gives the electric potential inside the sphere.

$$V(r < r_0) = V(r_0) - \int_{r_0}^r \frac{Qr}{4\pi\varepsilon_0 r_0^3} dr = \frac{Q}{4\pi\varepsilon_0 r_0} - \left.\frac{Qr^2}{8\pi\varepsilon_0 r_0^3}\right|_{r_0}^r = \boxed{\frac{Q}{8\pi\varepsilon_0 r_0}\left(3 - \frac{r^2}{r_0^2}\right)} \tag{16}$$

Alternative solution:

The potential can be obtained as an indefinite integral with the integration constant chosen such to make the potential continuous at the surface of the sphere:

$$V(r < r_0) = -\int \frac{Qr}{4\pi\varepsilon_0 r_0^3} dr = -\frac{Qr^2}{8\pi\varepsilon_0 r_0^3} + C \tag{17}$$

$$V(r = r_0) = -\frac{Q}{8\pi\varepsilon_0 r_0} + C = \frac{Q}{4\pi\varepsilon_0 r_0} \longrightarrow C = \frac{3Q}{8\pi\varepsilon_0 r_0}, \boxed{V(r < r_0) = \frac{Q}{8\pi\varepsilon_0 r_0}\left(3 - \frac{r^2}{r_0^2}\right)} \tag{18}$$

(c) To plot, we first calculate $V_0 = V(r = r_0) = \frac{Q}{4\pi\varepsilon_0 r_0}$ and $E_0 = E(r = r_0) = \frac{Q}{4\pi\varepsilon_0 r_0^2}$. Then we plot V/V_0 and E/E_0 as functions of r/r_0.

For $r < r_0$:

$$\frac{V}{V_0} = \frac{\frac{Q}{8\pi\varepsilon_0 r_0}\left(3 - \frac{r^2}{r_0^2}\right)}{\frac{Q}{4\pi\varepsilon_0 r_0}} = \frac{1}{2}\left(3 - \frac{r^2}{r_0^2}\right) \tag{19}$$

$$\frac{E}{E_0} = \frac{\frac{Qr}{4\pi\varepsilon_0 r_0^3}}{\frac{Q}{4\pi\varepsilon_0 r_0^2}} = \frac{r}{r_0} \tag{20}$$

For $r > r_0$:

$$\frac{V}{V_0} = \frac{\frac{Q}{4\pi\varepsilon_0 r}}{\frac{Q}{4\pi\varepsilon_0 r_0}} = \frac{r_0}{r} = \left(\frac{r}{r_0}\right)^{-1} \tag{21}$$

$$\frac{E}{E_0} = \frac{\frac{Q}{4\pi\varepsilon_0 r^2}}{\frac{Q}{4\pi\varepsilon_0 r_0^2}} = \frac{r_0^2}{r^2} = \left(\frac{r}{r_0}\right)^{-2} \tag{22}$$

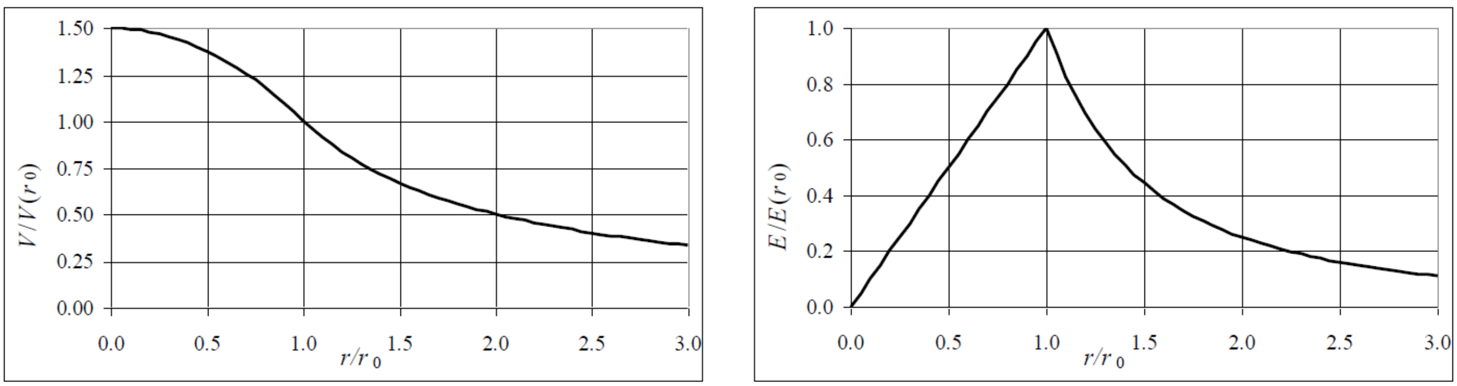

Figure 2: Electric potential and electric field dependence on the distance r.

Exercise 3 :

An electric transmission line in a house consists in two cables of a length l and radius $R = 1$ mm, spaced by a distance $d = 10$ cm. Suppose that $l \gg d \gg R$. Compute the capacitance per unit length of the transmission line. The capacity proves to have a negative effect in the transmission of the signal. What can you do to reduce this effect?

- **Hint**: remember that the capacitance is given by $C = \frac{Q}{\Delta V}$

Solution 3 :

The definition of capacity seen in the course is:

$$C = \frac{Q}{V} \tag{23}$$

To find the capacity C of this arrangement of two cables we need to find the potential difference between the two cables that carry charge $+Q$ and $-Q$ respectively.

We start by calculating the electric field generated by this distribution of charges using the Gauss law, then we use this result to obtain the potential difference V.

$$\oint_S \overrightarrow{E} \cdot d\overrightarrow{\sigma} = \frac{Q}{\varepsilon_0} \tag{24}$$

By symmetry we expect the field $\overrightarrow{E}$ to be radial. Taking S as a cylinder of radius r and length l concentric with the cable we find :

$$E(r)2\pi rl = \frac{q}{\varepsilon_0} \tag{25}$$

with q as the charge enclosed by S. Only the lateral surface of the cylinder contributes to the integral, because $\overrightarrow{E}$ and $d\overrightarrow{\sigma}$ are perpendicular on the cylinder faces.

The electric field generated by the positively charged cable can be expressed as:

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{26}$$

where $\lambda = \frac{q}{l}$ is the charge density per unit length
The same method is applied to find the electric field for the second cable and we obtain the total electric field (see Fig.3):

$$E_{tot}(x) = \frac{\lambda}{2\pi\varepsilon_0 x} + \frac{\lambda}{2\pi\varepsilon_0(d-x)}. \tag{27}$$

(To find the electric field of the first cable, we fixed the origin of the x coordinate at the center of the cable. In calculating the electric field of the second cable, this convention must be maintained).

With this expression we calculate the potential difference :

$$\begin{aligned}
\Delta V = \int_R^{d-R} \overrightarrow{E}\, dl &= \frac{\lambda}{2\pi\varepsilon_0} \int_R^{d-R} \left[\frac{1}{x} + \frac{1}{d-x}\right] dx & (28)\\
&= \frac{\lambda}{2\pi\varepsilon_0}[ln(x) - ln(d-x)]\Big|_R^{d-R} & (29)\\
&= \frac{\lambda}{\pi\varepsilon_0}[ln(d-R) - ln(R)] & (30)\\
&\approx \frac{\lambda}{\pi\varepsilon_0}[ln(d) - ln(R)] & (31)
\end{aligned}$$

$$\boxed{\Delta V \approx \frac{Q}{\pi\varepsilon_0 l} ln\left(\frac{d}{R}\right)} \tag{32}$$

Finally we find the capacity per unit length:

$$\frac{C}{l} = \frac{Q}{\Delta V l} \approx \frac{\pi\varepsilon_0}{ln\left(\frac{d}{R}\right)} \approx 6.04 \;\; pF/m \tag{33}$$

To reduce the capacity per unit length one could **increase** the **distance between the cables** d or **reduce** the **radius** of the cables R.

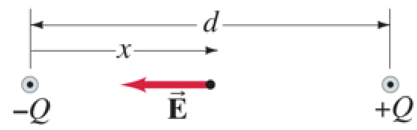

Figure 3: Electric field between the two cables.

Exercise 4 :

Refer to Fig.2 and consider the three following cases.

(a) An insulating spherical shell, centered at the origin O of the Cartesian axes, is uniformly charged (with internal radius a, external radius b and total charge $+Q$). Calculate at any point in space the electric potential $V(\vec{r})$ and the electric field $\vec{E}(\vec{r})$. Graphically represent the functions $V(\vec{r})$ and the radial component of $\vec{E}(\vec{r})$.

(b) A conductive spherical shell, centered at the origin O of the Cartesian axes, is electrically neutral and floating (internal radius a and external radius b). A $+Q$ charge is placed in the center of the shell (in O). Calculate at any point in space the electric potential $V(\vec{r})$ and the electric field $\vec{E}(\vec{r})$. Graphically represent functions $V(\vec{r})$ and the radial component of $\vec{E}(\vec{r})$.

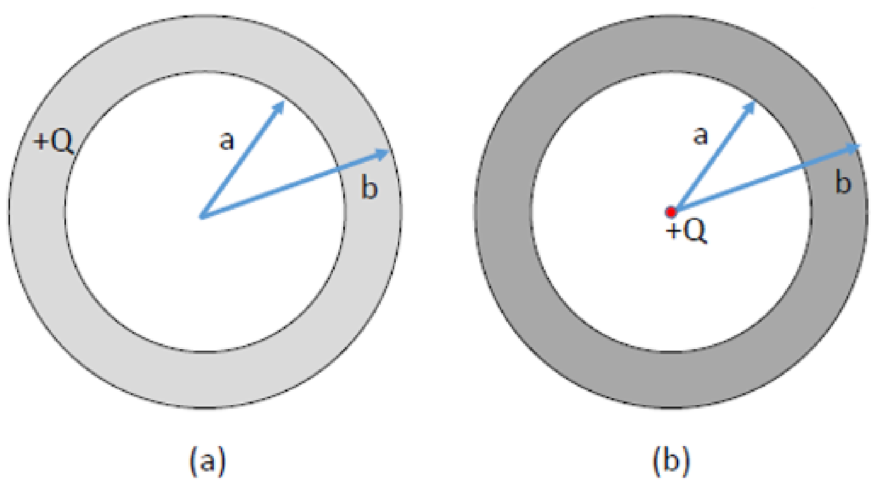

Figure 4: a) Insulating spherical shell uniformly charged with a total charge +Q. b) Neutral conductive spherical shell with a +Q charge in the centre. The shell is floating, i.e. electrically disconnected from the environment.

- **Hint 1**: use Gauss's law to compute the electric field first and then the potential by integrating the electric field.

Solution 4 :

The symmetry of the problem suggests that the electric field is directed radially which means that it only depends on the distance of the point with respect to the origin O (center of the shells). Hence we can write $\overrightarrow{E}(\overrightarrow{r}) = E(r)\overrightarrow{e_r}$.

(a) To calculate the electric field Gauss law is applied:

For r<a:

$$4\pi r^2 E(r) = \frac{Q_{enc}}{\varepsilon_0} \longrightarrow E(r) = 0 \tag{34}$$

because there is no charge inside the shell.

For a<r<b:

$$4\pi r^2 E(r) = \frac{4\pi}{3\varepsilon_0}(r^3 - a^3)\rho \tag{35}$$

where ρ is the charge density:

$$\rho = \frac{Q}{\mathcal{V}} = \frac{3Q}{4\pi(b^3 - a^3)}, \tag{36}$$

where $\mathcal{V}$ is the volume, and hence we find:

$$E(r) = \frac{\rho}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right) \tag{37}$$

For r>b:

$$4\pi r^2 E(r) = \frac{Q_{enc}}{\varepsilon_0} \longrightarrow E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \tag{38}$$

The electric potential is found by integrating the electric field with respect to r:

For r<a: $V(r) = D$

For r>b: $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ (by taking $V(r=\infty) = 0$)

For a<r<b: $V(r) = -\frac{\rho}{3\varepsilon_0}\left(\frac{r^2}{2} + \frac{a^3}{r}\right) + C$

For the potential to be a continuous function in $r = b$ and $r = a$ we need to determine C and D:

For $r = b$:

$$V(b) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{b} = -\frac{\rho}{3\varepsilon_0}\left(\frac{b^2}{2} + \frac{a^3}{b}\right) + C \tag{39}$$

we find that:

$$C = \frac{\rho}{2\varepsilon_0}b^2 \tag{40}$$

For r=a:

$$-\frac{\rho}{3\varepsilon_0}\left(\frac{a^2}{2}+\frac{a^3}{a}\right)+C=D \quad (41)$$

after plugging in the expression for C, we find that:

$$D=\frac{\rho}{2\varepsilon_0}(b^2-a^2) \quad (42)$$

Finally:

$V(r)=\frac{\rho}{2\varepsilon_0}(b^2-a^2)$ for $r<a$

$V(r)=-\frac{\rho}{3\varepsilon_0}\left(\frac{r^2}{2}+\frac{a^3}{r}\right)+\frac{\rho}{2\varepsilon_0}b^2$ for $a<r<b$

$V(r)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ for $r>b$

Notice that the electric field is continuous. This is because the volume charge density is finite everywhere.

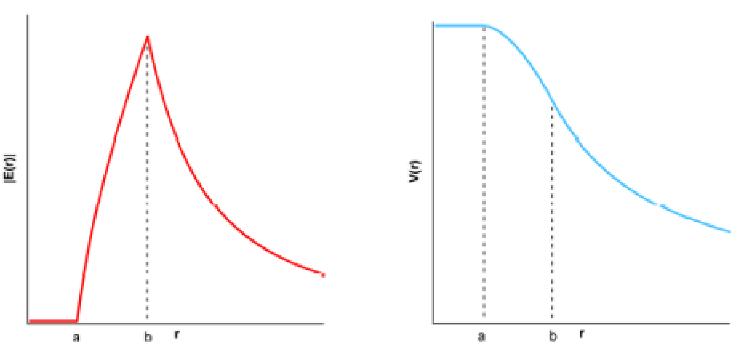

Figure 5: Electric field and electric potential as a function of the distance r in the case of an insulating spherical shell.

(b) Assuming static conditions, the electric field $\overrightarrow{E}$ is 0 inside the conductor. By using the Gauss Law, we get:

$E(r)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$ for $r<a$

$E(r)=0$ for $a<r<b$

$E(r)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$ for $r>b$

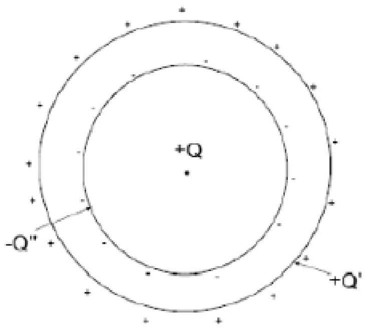

Figure 6: Charge distribution of a conductive neutral sphere with a $+Q$ charge in the middle.

To find the electric potential we integrate the expressions found for the electric field and obtain:

$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + D$ for $r < a$

$V(r) = C$ for $a < r < b$

$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$ for r>b and assuming $V(\infty) = 0$

And we find that C and D are equal to:

$$C = \frac{Q}{4\pi\varepsilon_0}\frac{1}{b} \tag{43}$$

$$D = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) \tag{44}$$

Notice the discontinuities of the electric field at $r = a$ and $r = b$ with $E(a^-) \neq E(a^+)$, $E(b^-) \neq E(b^+)$, $\Delta E(r = a) = E(a^-) - E(a^+) = \frac{\sigma_-}{\varepsilon_0}$, $\Delta E(r = b) = E(b^-) - E(b^+) = \frac{\sigma_+}{\varepsilon_0}$. This is due to the surface charge densities σ_+ and σ_- which corresponds to infinite volume charge densities.

$$\frac{\Delta E(r = a)}{\Delta E(r = b)} = \frac{\sigma_-}{\sigma_+} = \frac{Q}{a^2}\frac{b^2}{Q} = \frac{b^2}{a^2} \tag{45}$$

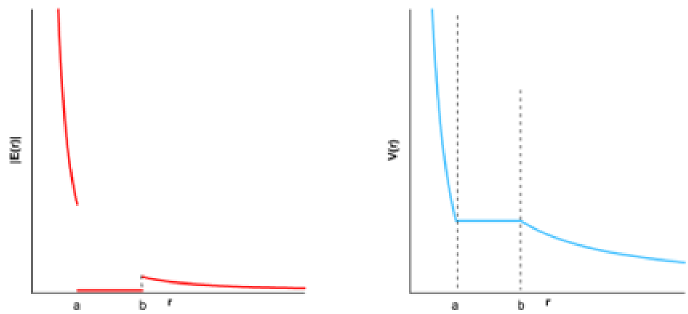

Figure 7: Electric field and electric potential as a function of the distance r in the case of a conductive spherical shell.

Exercise 5 :

Diagnostic imaging techniques, such as for example ultrasounds, use coaxial cables to transmit instruments data. A coaxial cable with a length h is made up of a conductive cylinder with radius R_1 inside a second empty cylinder of radius R_2, separated by a dielectric, as shown in Fig.2. We consider here a coaxial cable with air as a dielectric. An advantage of this kind of cables is that the external electric noise is reduced, because the outer cylinder acts as a Faraday cage which shields the inner conductor. On the other hand, this type of cable acts as a cylindrical capacitor. A huge capacitance can result in a delay in the transmission, which can cause interference of the signal. It is also important that the capacitances are low for diagnostic imaging techniques because the images are high resolution.

- **Hint**: remember that the capacitance is given by $C = \frac{Q}{\Delta V}$

1. Find an expression for the capacitance per unit length of such a coaxial cable (consider $h \gg R_2$).

2. A coaxial cable factory propose you two cables for your diagnostic ultrasounds. The first cable has dimensions $R_1 = 0.25$ mm and $R_2 = 0.76$ mm. The second $R_1 = 0.08$ mm and $R_2 = 1.20$ mm. Which one do you choose?

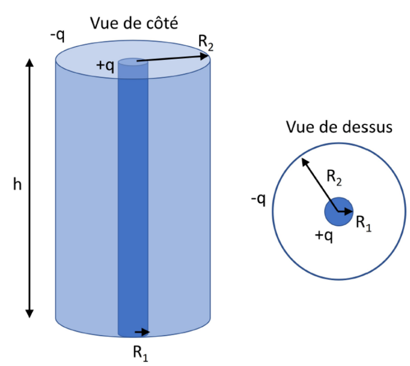

Figure 8: Schematic of a coaxial cable. Side and top view.

Solution 5 :

1. For a capacitor with charge q and potential difference ΔV, the capacity is defined as:

$$C = \frac{q}{\Delta V} \tag{46}$$

We need to find the expression for the potential difference between the two cylinders. We start by applying the Gauss law:

$$\oint_S \overrightarrow{E} \cdot d\overrightarrow{\sigma} = \frac{Q_{enc}}{\varepsilon_0} \tag{47}$$

We consider a cylindrical Gaussian surface between the two cylinders. $Q_{enc} = q$ and $dS = 2\pi r dy$ (between $y = 0$ and $y = h$). By symmetry $\overrightarrow{E} = \overrightarrow{E}(r)$ and hence, we can write:

$$E(r) = \frac{q}{2\pi\varepsilon_0 rh} \tag{48}$$

And the potential difference is:

$$\Delta V = V_1 - V_2 = -\int_{R_2}^{R_1} \frac{q}{2\pi\varepsilon_0 h}\frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 h} ln\left(\frac{R_2}{R_1}\right) \tag{49}$$

Finally, the capacity per unit length is:

$$\frac{C}{h} = \frac{q}{h\Delta V} = \boxed{\frac{2\pi\varepsilon_0}{ln\left(\frac{R_2}{R_1}\right)}} \tag{50}$$

2. By using the equation 50 found in question 1., the capacity by unit length for the first coaxial cable is $5.00 \cdot 10^{-11}$F and for the second it is $2.05 \cdot 10^{-11}$F. In conclusion, the second coaxial cable is more favorable for a system of diagnostic imaging.