General Physics: Electromagnetism, Correction 3

Exercise 1 :

Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$.

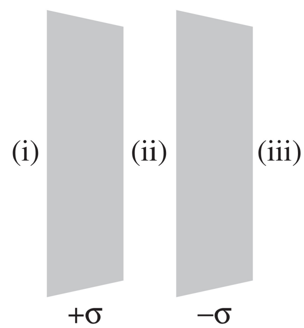

Figure 1: Two infinite parallel planes carrying opposite uniform charges.

Find the field in each of the three regions:

(i) To the left of both;

(ii) Between them;

(iii) To the right of both.

Solution 1 :

The left plate produces a field $\dfrac{\sigma}{2\epsilon_0}$ which points ***away*** from it - to the left in region **(i)** and to the right in regions **(ii)** and **(iii)**. The right plate being negatively charged, produces a field $\dfrac{-\sigma}{2\epsilon_0}$ pointing ***toward*** it - to the right in regions **(i)** and **(ii)** and to the left in region **(iii)**. The two fields cancel in regions **(i)** and **(iii)**; they add up in region **(ii)**.

Conclusion: The field between the plate is $\dfrac{\sigma}{\epsilon_0}$ and it points to the right, elsewhere the field is zero.

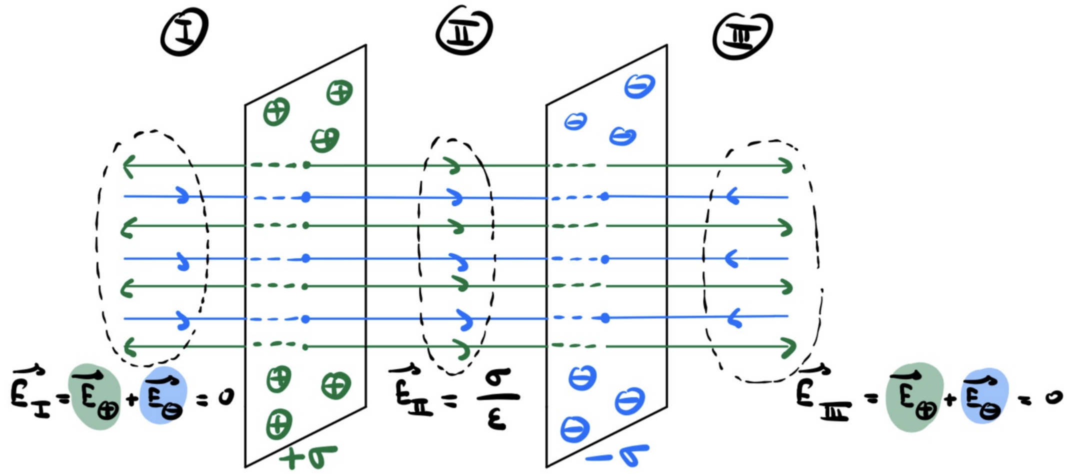

Figure 2: Two infinite parallel planes carrying opposite uniform charges. The electric field points away from a positively charged surface, whereas it points inside a negatively charged surface.

Exercise 2 :

We assume an electron beam is a stationary uniform charge distribution in cylindric form with radius a and infinite length.

(a) Find the electric field at a distance r from the beam center for $r > a$ and $r < a$. Assume a line charge density of λ of the electron beam.

(b) What is the force on an electron in the beam at a distance r from the beam axis if you assume a volume charge density of n electrons per unit volume V?

Solution 2 :

(a) The infinite length cylinder has translational symmetry along the longitudinal direction. We thus use the same geometry for the Gaussian surface as explained in the lecture; a cylinder with length l and radius r. We only need to consider the electrons at positions $\leq r$ because only these are enclosed by our Gaussian surface. The flux through this surface becomes:

$$\Phi = \oint \vec{E}(r) \cdot d\vec{A} = \frac{Q(r)}{\varepsilon_0} \tag{1}$$

The flux through the two disk-shaped sides is 0, since the E-field is orthogonal to the surface normal vector. What remains is the flux through the lateral surface.

$$\Rightarrow 2\pi r l E(r) = \frac{Q(r)}{\varepsilon_0} \tag{2}$$

$$\Rightarrow E(r) = \frac{Q(r)}{2\pi\varepsilon_0 r l} \tag{3}$$

In order to find the total charge enclosed by the Gaussian cylinder, we need to remember that when $r < a$, we are not enclosing the full beam but only a fraction of it. Since we know that the charge is uniformly distributed within the beam, we can find the enclosed charge by multiplying the total charge $Q = \lambda l$ of a beam section of length l with the ratio between the enclosed volume $V_{\text{encl}} = \pi r^2 l$ and the total volume of the section $V_{\text{tot}} = \pi a^2 l$. Therefore the enclosed charge is given by:

$$Q(r) = \begin{cases} \lambda \frac{r^2}{a^2} l, & r \leq a \\ \lambda l, & r > a \end{cases}, \tag{4}$$

hence

$$E(r) = \begin{cases} \frac{\lambda r}{2\pi\varepsilon_0 a^2}, & r \leq a \\ \frac{\lambda}{2\pi\varepsilon_0 r}, & r > a \end{cases} \tag{5}$$

The direction of the electric field points toward the center of the beam.

(b) Given that ne is the volume charge density ρ, the corresponding line charge density becomes $\lambda = \rho\pi a^2 = \pi a^2 ne$. We can thus replace λ in equation (5) and write the electric field as

$$E(r) = \begin{cases} \frac{ner}{2\varepsilon_0}, & r \leq a \\ \frac{nea^2}{2\varepsilon_0 r}, & r > a \end{cases} \tag{6}$$

The magnitude of the force $F = |\overrightarrow{F}| = |q\overrightarrow{E}|$ on an electron with charge $q = -e$ then becomes

$$F(r) = \begin{cases} \frac{ne^2 r}{2\varepsilon_0}, & r \leq a \\ \frac{ne^2 a^2}{2\varepsilon_0 r}, & r > a \end{cases} \tag{7}$$

Due to the negative charge of the electron, the direction of the force points away from the central axis of the beam, opposite of the direction of the electric field.

Exercise 3 :

A sphere of radius r_0 carries a volume charge density ρ_E (see Figure 3). A spherical cavity of radius $r_0/2$ is then scooped out and left empty, as shown.

(a) What is the magnitude and direction of the electric field at point A?

(b) What is the direction and magnitude of the electric field at point B?

Points A and C are at the centers of the respective spheres.
Hint: in order to take a symmetric Gaussian surface consider two spheres: a big one centered in A, with radius r_0 and with volume charge density ρ_E and a second smaller one centered in C, with radius $r_0/2$ and with volume charge density $-\rho_E$.

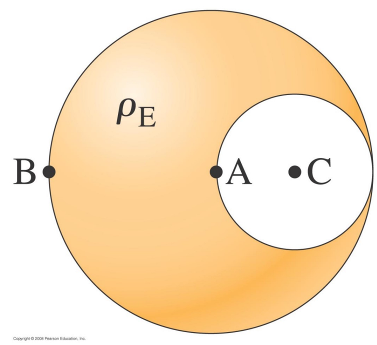

Figure 3: A charged sphere of radius r_0 containing a cavity of radius $r_0/2$.

Solution 3 :

Consider this sphere as a combination of two spheres. **Sphere 1** is a solid sphere of r_0 and charge density $+\boldsymbol{\rho_E}$ centered at A and **Sphere 2** is a second sphere of radius $\frac{r_0}{2}$ and charge density $-\boldsymbol{\rho_E}$ centered at C.

(a) The electric field at $\boldsymbol{A}$ will have **zero contribution** from **Sphere 1** due to its symmetry about point A. **Sphere 2**, on the other hand, contributes to the electric field. The electric field is then calculated by creating a Gaussian surface centered at point C with radius $\frac{r_0}{2}$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow E \cdot 4\pi(\frac{1}{2}r_0)^2 = \frac{(-\rho_E)\frac{4}{3}\pi(\frac{1}{2}r_0)^3}{\varepsilon_0} \rightarrow \boxed{E = -\frac{\rho_E r_0}{6\varepsilon_0}} \quad (8)$$

Since the electric field points into the Gaussian surface (hence the negative sign) the electric field at point $\boldsymbol{A}$ points to the **right** (i.e. towards the center of Sphere 2)

(b) At point $\boldsymbol{B}$ the electric field will be the sum of the the electric fields from each sphere. The electric field from **Sphere 1** is calculated using the Gaussian surface of radius $\boldsymbol{r_0}$ centered at $\boldsymbol{A}$.

$$\oint \vec{E_1} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow E_1 \cdot 4\pi r_0^2 = \frac{(+\rho_E)\frac{4}{3}\pi r_0^3}{\varepsilon_0} \rightarrow E_1 = \frac{\rho_E r_0}{3\varepsilon_0} \quad (9)$$

At point $\boldsymbol{B}$ the field from **Sphere 1** points to the **left**.

The electric field from **Sphere 2** is calculated using a Gaussian surface centered at $\boldsymbol{C}$ of radius $\frac{3}{2}\boldsymbol{r_0}$ (i.e. the distance of B to C).

$$\oint \overrightarrow{E_2} \cdot d\overrightarrow{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow E_2 \cdot 4\pi(\frac{3}{2}r_0)^2 = \frac{(-\rho_E)\frac{4}{3}\pi(\frac{1}{2}r_0)^3}{\varepsilon_0} \rightarrow E_2 = -\frac{\rho_E r_0}{54\varepsilon_0} \tag{10}$$

At point $\boldsymbol{B}$, the electric field from **Sphere 2** points toward the **right**.

The **net electric field** is the **sum** of these two fields.

$$E = E_1 + E_2 = \frac{\rho_E r_0}{3\varepsilon_0} + (-\frac{\rho_E r_0}{54\varepsilon_0}) = \boxed{\frac{17\rho_E r_0}{54\varepsilon_0}} \tag{11}$$

The net field points to the left.

Exercise 4 :

A solid insulating sphere of radius a carries a net positive charge Q uniformly distributed throughout its volume. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-2Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in Figure 4 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

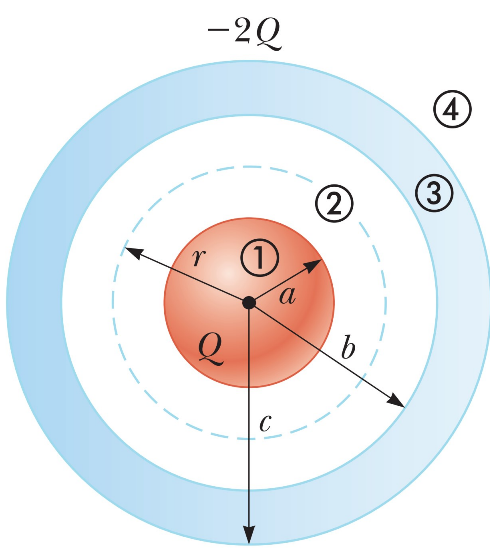

Figure 4: An insulating sphere of radius a and carrying a charge Q surrounded by a conducting spherical shell carrying a charge $-2Q$.

Solution 4 :

The charge is distributed uniformly throughout the sphere and the charge on the conducting shell distributes itself uniformly on the surfaces. Hence, the system has spherical symmetry and the Gauss's law can be applied by using spherical Gauss surfaces to find the electric field in the different regions.

1. The electric field **inside** a **conducting shell** is **zero**. Therefore for region ① the electric field is due to the sphere.

$$E_1 = k_e \frac{Q}{a^3} r \quad (for\ r < a) \tag{12}$$

2. The electric field in region ② is only due to the sphere and not the shell (because r<b).

$$E_2 = k_e \frac{Q}{r^2} \quad (for\ a < r < b) \tag{13}$$

3. The electric field in region ③ is **zero** because the spherical shell is a conductor in equilibrium.

$$E_3 = 0 \quad (for\ b < r < c) \tag{14}$$

4. The electric field in region ④ is due to the sphere and conducting shell. The total charge that the Gaussian surface of the region ④ surrounds is a total charge $q_{in} = q_{sphere} + q_{shell} = Q + (-2Q) = -Q$. Therefore the electric field in region ④ would be the same as if the field was generated by a sphere of charge -Q.

$$E_4 = -k_e \frac{Q}{r^2} \quad (for\ r > c) \tag{15}$$

Exercise 5 :

A long, straight wire is surrounded by a hollow metal cylinder whose axis coincides with that of the wire. The wire has a charge per unit length of λ, and the cylinder has a net charge per unit length of 2λ. From this information, use Gauss's law to find:

(a) The charge per unit length on the inner surface of the cylinder;

(b) The charge per unit length on the outer surface of the cylinder;

(c) The electric field outside the cylinder a distance r from the axis.

Solution 5 :

(a) Inner surface: Consider a cylindrical Gaussian surface of arbitrary length l within the metal cylinder. The electric field E inside the conducting shell is zero, therefore the total charge inside the Gaussian surface must also be zero:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \to 0 = \frac{(\lambda + \lambda_{inner})l}{\varepsilon_0} \to \boxed{\lambda_{inner} = -\lambda} \tag{16}$$

In this case for the electric field to be zero inside the metal cylinder. The charge in the inner surface of the cylinder has to cancel out the charge of the wire.

(b) Outer surface: Consider a cylindrical Gaussian surface of arbitrary length l outside the metal. The total charge within the Gaussian surface is:

$$q_{wire} + q_{cylinder} = q_{wire} + (q_{outersurface} + q_{innersurface}) \tag{17}$$

$$\lambda l + 2\lambda l = \lambda l + (\lambda_{outer} l + (-\lambda l)) \to \boxed{\lambda_{outer} = 3\lambda} \tag{18}$$

(c) Gauss's law applied to a cylindrical surface:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \tag{19}$$

$$E \cdot 2\pi r l = \frac{3\lambda l}{\varepsilon_0} \to E = 2\frac{3\lambda}{4\pi\varepsilon_0 r} = \boxed{6k_e\frac{\lambda}{r}} \tag{20}$$

The field points radially outward.