# General Physics: Electromagnetism, Solution 2

## Exercise 1 :

In the Bohr theory of the hydrogen atom, an electron moves in a circular orbit about a proton, where the radius of the orbit is $a_0 = 5.29 \times 10^{-11}$ m. The proton has charge $e = 1.60 \times 10^{-19}$ C and the electron has charge $-e$ and mass $m_\mathrm{e} = 9.11 \times 10^{-31}$ kg.

1. Find the magnitude of the electric force exerted on each particle.
2. If this force causes the centripetal acceleration of the electron, what is the speed of the electron?

## Solution 1 :

1. The magnitude of the force exerced on each particle is:

$$|F| = \frac{k_e e^2}{a_0^2} = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}} \frac{(1.60 \times 10^{-19}\,\mathrm{C})^2}{(0.529 \times 10^{-10}\,\mathrm{m})^2} = 8.22 \times 10^{-8}\,\mathrm{N}. \tag{1}$$

2. The centripetal force is defined as: $F = \frac{mv^2}{r}$ from which we can find:

$$v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{8.22 \times 10^{-8}\,\mathrm{N} 0.529 \times 10^{-10}\,\mathrm{m}}{9.11 \times 10^{-31}\,\mathrm{kg}}} = 2.19 \times 10^6\,\mathrm{m\,s^{-1}}. \tag{2}$$

## Exercise 2 :

Charges $q_1$ and $q_2$ are located on the $x$ axis at distances $a$ and $b$ respectively from the origin, as shown in the next figure.

1. Find the components of the net electric field at the point $P$, which is at position $(0, y)$.
2. Evaluate the electric field at point $P$ in the special case that $|q_1| = |q_2|$ and $a = b$.
3. Still for the special case of problem 2., find the electric field due to the electric dipole when point $P$ is at distance $y \gg a$ from the origin.

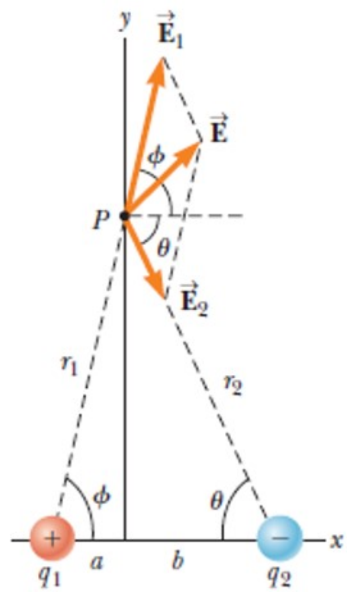


Figure 1: Positions of the charges $q_1$ and $q_2$ and of a field point $P$. The total electric field $\boldsymbol{E}$ at $P$ equals the vector sum $\boldsymbol{E}_1 + \boldsymbol{E}_2$, where $\boldsymbol{E}_1$ is the field due to the positive charge $q_1$ and $\boldsymbol{E}_2$ is the field due to the negative charge $q_2$.

## Solution 2 :

1. Here we make use of the superposition principle and calculate the effect of charges $q_1$ and $q_2$ individually and then add them together:

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 \tag{3}$$

For the individual field contributions, we get

$$\boldsymbol{E}_1 = k\frac{q_1}{|\boldsymbol{r}_1|^2}\hat{\boldsymbol{r}}_1 = k\frac{q_1}{a^2 + y^2}\hat{\boldsymbol{r}}_1 \tag{4}$$

$$\boldsymbol{E}_2 = k\frac{q_2}{|\boldsymbol{r}_2|^2}\hat{\boldsymbol{r}}_2 = k\frac{q_2}{b^2 + y^2}\hat{\boldsymbol{r}}_2, \tag{5}$$

where $\hat{\boldsymbol{r}}_1 = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$ and $\hat{\boldsymbol{r}}_2 = -\hat{\boldsymbol{x}}\cos\theta + \hat{\boldsymbol{y}}\sin\theta$ are the displacement vectors from charges $q_1$ and $q_2$ to $P$. We can now sum up the contributions of the two charges. For clarity, we write the $x$ and $y$ components of $\boldsymbol{E}$ separately:

$$E_x = E_{1x} + E_{2x} = k\frac{q_1}{a^2+y^2}\cos\phi - k\frac{q_2}{b^2+y^2}\cos\theta \tag{6}$$

$$= k\frac{|q_1|}{a^2+y^2}\cos\phi + k\frac{|q_2|}{b^2+y^2}\cos\theta \tag{7}$$

$$E_y = E_{1y} + E_{2y} = k\frac{q_1}{a^2+y^2}\sin\phi + k\frac{q_2}{b^2+y^2}\sin\theta \tag{8}$$

$$= k\frac{|q_1|}{a^2+y^2}\sin\phi - k\frac{|q_2|}{b^2+y^2}\sin\theta. \tag{9}$$

Keep in mind, that in Equations (7) and (9), the sign of second term changes because $q_2$ is negative.

2. In the case where $|q_1| = |q_2| = q$ and $a = b$, due to symmetry we can also see that $\phi = \theta$ and Equations (7) and (9) become much simpler, such that

$$E_x = k\frac{q}{a^2+y^2}\cos\theta + k\frac{q}{a^2+y^2}\cos\theta = 2k\frac{q}{a^2+y^2}\cos\theta \tag{10}$$

$$E_y = k\frac{q}{a^2+y^2}\sin\theta - k\frac{q}{a^2+y^2}\sin\theta = 0. \tag{11}$$

With $\cos\theta = a/r = a/(a^2+y^2)^{1/2}$, we get

$$E_x = k\frac{2aq}{(a^2+y^2)^{3/2}}. \tag{12}$$

3. At large distance, where $y \gg a$, we can neglect $a$ in the denominator of Equation (12) and get

$$E_x = k\frac{2aq}{y^3}. \tag{13}$$

In contrast to the $1/r^2$ decay of the electric field of a point charge, here we have a $1/r^3$ dependence. At large distances, the two opposite charges almost neutralize each other, which explains the missing $1/r^2$ dependence. What remains is the electric dipole field, which decays as $1/r^3$. In fact, the $1/r^3$ dependence of $\boldsymbol{E}$ is also obtained for a distant point along the $x$-axis.

## Exercise 3 :

A ring of radius $a$ carries a uniformly distributed positive total charge $Q$.

1. Calculate the electric field due to the ring at a point $P$ lying at a distance $x$ from its center along the central axis perpendicular to the plane of the ring (see next figure).

2. Suppose a negative charge is placed at the center of the ring and displaced slightly by a distance $x \ll a$ along the axis. When the charge is released, what type of motion does it exhibit?

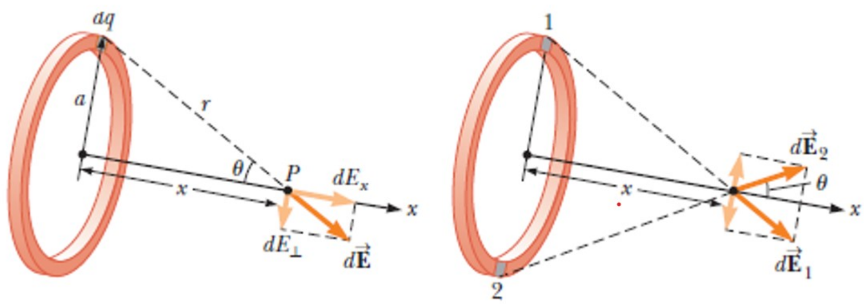


Figure 2: Ring with uniform positive charge $Q$ and field point $P$ at a distance $x$ on the central axis of the ring. The infinitesimal electric field $d\boldsymbol{E}$ produced by an elementary charge $dq$ on the ring in the point $P$ can be divided in a component along the central axis $dE_x$ and in a component lying on the ring's plane $dE_\perp$. The total electric field in the point $P$ is given by the sum of the infinitesimal electric fields produced by all the elementary charges on the ring.

## Solution 3 :

1. Observation:
   If we consider two charges, called $q_1$ and $q_2$, at the opposite side of the ring the total field generated by them along the direction perpendicular to the central axis perpendicular to the ring plane ($E_\perp$) is zero. That is because the contributions $E_{1,\perp}$ and $E_{2,\perp}$ have opposite sign. This consideration is valid for any couple of charges on the ring. So, in the end we are just interested in the $\hat{x}$ component.
   The ring is a continuous of infinitesimal charges $dq$, we compute the infinitesimal $d\vec{E}$ generated by a single infinitesimal charge $dq$ and then we sum up all the contributions, which means, for a continuous charge distribution, integrating along the ring.
$$dE_x = k_e \frac{dq}{r^2}\cos\theta = k_e \frac{dq}{a^2+x^2}\cos\theta, \tag{14}$$
   with,
$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}}. \tag{15}$$
   Hence we get,
$$dE_x = k_e \frac{dq}{a^2+x^2}\frac{x}{\sqrt{a^2+x^2}} = \frac{k_e x}{\sqrt{(a^2+x^2)^3}}dq. \tag{16}$$
   The total electric field along the $\hat{x}$ component is
$$E_x = \int_0^Q dq k_e \frac{x}{(a^2+x^2)^{3/2}} = \frac{k_e x}{(a^2+x^2)^{3/2}}\int_0^Q dq = \frac{k_e x}{(a^2+x^2)^{3/2}}Q. \tag{17}$$
   Finally the total electric field is $\vec{E} = E_x\vec{x} = \frac{k_e x}{(a^2+x^2)^{3/2}}Q\hat{x}$. Notice that for $x \gg a$, i.e. far from the ring, as expected the expression becomes:
$$\vec{E} \approx \frac{k_e x Q}{x^3}\hat{x} = k_e \frac{Q}{x^2}\hat{x}, \tag{18}$$

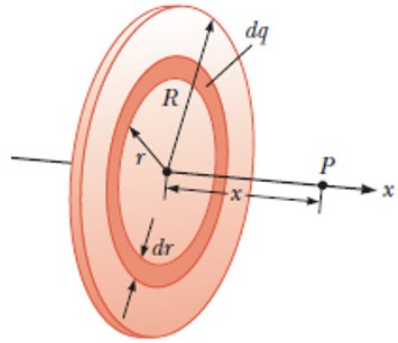


Figure 3: Disk of radius $R$ with uniform surface density $\sigma$. The elementary charge $dq$ is given by a circular crown of width $dr$ located at distance $r$ from the center. The field point is at distance $x$ on the central axis of the ring.

equivalent to the case of a point-like charge.

2. For the case where $x \ll a$, the field generated by the ring is:

$$E_x = \frac{k_e x Q}{(a^2 + x^2)^{3/2}} \approx \frac{k_e x Q}{a^3}. \tag{19}$$

The force exerted on the negative charge q is:

$$\vec{F} = -\dot{q}\vec{E} = -qE_x = -\frac{k_e q Q}{a^3} = -\text{const} \cdot x, \tag{20}$$

equivalent to a Hooke's like force, the motion of the particle will be harmonic, oscillating near the center of the ring.

## Exercise 4 :

A disk of radius $R$ has a uniform surface charge density $\sigma$.

1. Calculate the electric field at a point $P$ that lies at a distance $x$ from the disk's center along the central axis perpendicular to the plane of the disk (see next figure).

2. What happens if we let the radius of the disk grow so that the disk becomes an infinite plane of charge?

3. Show that the electric field at distances $x$ that are large compared with $R$ approaches that of a particle with charge $Q = \sigma\pi R^2$.

   Hint: Use the fact that $x/(x^2 + R^2)^{1/2} = (1 + R^2/x^2)^{-1/2}$ and the approximation for the binomial expansion $(1 + \delta)^n \approx 1 + n\delta$ when $\delta \ll 1$.

## Solution 4 :

1. Similarly to the previous exercise, here we can split the disk into infinitesimally thin rings and sum up the contribution of each ring to get the total electric field. Fortunately, we already calculated the electric field for a single ring in the previous exercise. Rewriting the previous result with the newly defined variables (see Figure 3) to get the electric field $dE_x$ generated by an infinitesimally thin ring of radius $r$:

$$dE_x = k_e \frac{x dq}{(r^2 + x^2)^{3/2}}, \tag{21}$$

where $dq$ is the total charge of the infinitesimal ring. In order to get the total electric field, we need to integrate this term over all rings on the disk, so we need to find a way to substitute $dq$ with a term that we can easily integrate. Knowing that the total charge $q$ of a uniformly charged disk with radius $r$ is given by $q = \sigma A = \sigma \pi r^2$, we can derive this expression with respect to $r$ to get the infinitesimal charge of a ring:

$$dq = (2\pi r \sigma) dr \tag{22}$$

We can then use this expression to integrate $dE_x$ to get the total electric field:

$$E_x = \int_Q k_e \frac{x dq}{(r^2 + x^2)^{3/2}} = \int_0^R k_e \frac{x(2\pi r \sigma) dr}{(r^2 + x^2)^{3/2}} \tag{23}$$

$$= k_e \pi x \sigma \int_0^R \frac{2r dr}{(r^2 + x^2)^{3/2}} = k_e \pi x \sigma \int_0^{R^2} (u + x^2)^{-3/2} du \tag{24}$$

$$= k_e \pi x \sigma \left[ -\frac{1}{1/2} (u + x^2)^{-1/2} \right]_0^{R^2} = k_e \pi x \sigma \left[ \frac{2}{\sqrt{x^2}} - \frac{2}{(R^2 + x^2)^{1/2}} \right] \tag{25}$$

$$= 2\pi k_e \sigma \left[ \frac{x}{|x|} - \frac{x}{(R^2 + x^2)^{1/2}} \right], \tag{26}$$

where we substitute $r$ with $u = r^2$, $dr$ with $du = 2r \cdot dr$ and $\frac{x}{|x|} = \frac{x}{\sqrt{x^2}}$. Note that we also have to substitute the boundaries of the integral accordingly.

2. In the limit where $x \ll R$ ($R \to \infty$), the above equation simplifies to

$$E_x = 2\pi \sigma k_e = 2\pi \sigma \frac{1}{4\pi \epsilon_0} = \frac{\sigma}{2\epsilon_0} \tag{27}$$

which corresponds to the electric field generated by an infinite plane uniformly charged (see Gauss theorem in the next lecture).

3. We can find the electric field in the the limit where $x >> R$ using the result of question 1

$$E_x = 2\pi \sigma k_e \left( 1 - \frac{x}{\sqrt{R^2 + x^2}} \right) = 2\pi \sigma k_1 \left( 1 - \frac{x}{x\sqrt{\frac{R}{x}^2 + 1}} \right) = 2\pi \sigma k_1 \left( 1 - \frac{1}{\sqrt{\frac{R}{x}^2 + 1}} \right) \tag{28}$$

In the limit of $\frac{R^2}{x^2} \ll 1$, we can expand $\left(1+\frac{R^2}{x^2}\right)^{-1/2} = 1 - \frac{R^2}{2x^2}$. Substituting above, we get

$$E_x = 2\pi\sigma k_e \frac{R^2}{2x^2} = \pi\sigma k_e \frac{R^2}{x^2} \tag{29}$$

with $\sigma = \frac{Q}{A} = \frac{Q}{\pi R^2}$

$$E_x = \pi\sigma k_e \frac{R^2}{x^2} = \pi \frac{Q}{\pi R^2} k_e \frac{R^2}{x^2} = k_e \frac{Q}{x^2} \tag{30}$$

which is simply the electric field generated by a point-like charge because the disk at long distance looks like a single charge $Q$.

## Exercise 5 :

A proton of mass $m_\mathrm{p} = 1.67 \times 10^{-27}$ kg moves at $4.50 \times 10^5$ m/s in horizontal direction. It enters a uniform vertical electric field with a magnitude of $9.60 \times 10^3$ N/C. Ignoring any gravitational effects, find:

1. the time interval required for the proton to travel 5.00 cm horizontally,
2. its vertical displacement during the time interval in which it travels 5.00 cm horizontally,
3. the horizontal and vertical components of its velocity after it has traveled 5.00 cm horizontally.

## Solution 5 :

1. In the horizontal direction, the speed is constant since there is no horizontal force applied on the proton (the field is only along the vertical direction). Consequently, the time interval required to traver 5 cm horizontally is

$$d = v_x t \to t = \frac{d}{v_x} = 111 \text{ ns} \tag{31}$$

2. Since the vertical field, the proton feels a vertical force which results in a vertical and constant acceleration. This acceleration is given by

$$F_y = m a_y \to a_y = \frac{F_y}{m} = \frac{qE}{m} = 9.21 \times 10^{11}\,\mathrm{m\,s^{-2}} \tag{32}$$

Under a constant acceleration, the displacement is given by

$$y = \frac{1}{2} a_y t^2 + v_{y,i} t + y_0 \tag{33}$$

with initial velocity $v_{y,i} = 0$ and $y_0 = 0$, the displacement is simply

$$y = \frac{1}{2} a_y t^2 = 5.68\,\mathrm{mm} \tag{34}$$

3. Without any horizontal force, the velocity along the $x$ direction is constant. The velocity along the y direction is given by

$$v_y = v_{y,i} + a_y t \tag{35}$$

with initial velocity $v_{y,i} = 0$, velocity is simply

$$v_y = a_y t = 1.02 \times 10^5\,\mathrm{m\,s^{-1}} \tag{36}$$

Consequently,

$$\vec{v} = \left(4.5 \times 10^5 \hat{x} + 1.02 \times 10^5 \hat{y}\right)\mathrm{m\,s^{-1}} \tag{37}$$