General Physics: Electromagnetism, Correction 1

Refresher: Vector Algebra

This section is adapted from the first chapter of *Introduction to Electrodynamics* by **David J. Griffiths**.

Vector operations

In order to describe quantities which have *direction* and *magnitude*, such as displacements, velocity, acceleration, force, etc... one is in need of **vectors** obeying their own arithmetic. In these exercises we will denote "vector A" as $\boldsymbol{A}$ and its magnitude as $|\boldsymbol{A}|$ or also as A.

We can define four vectors operations:

1. **Addition of two vectors.**

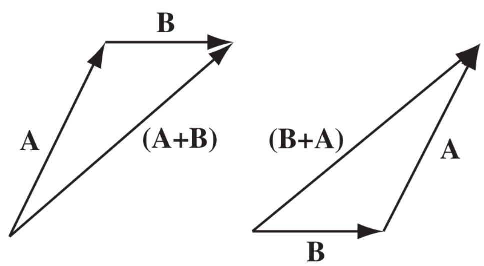

Figure 1: Addition of two vectors

Place the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$; the sum $\boldsymbol{A}+\boldsymbol{B}$ is the vector from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$ (Fig. 1).

- Addition is *commutative*:

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}. \tag{1}$$

- Addition is *associative*:

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}). \tag{2}$$

2. **Multiplication by a scalar.**
Multiplication of a vector by a scalar a multiplies the *magnitude* but leaves the direction unchanged. Scalar multiplication is *distributive*.

$$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B}. \tag{3}$$

3. **Dot product of two vectors.**
The dot product of two vectors is defined as,

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv AB\cos(\theta), \tag{4}$$

where θ is the angle they form when placed tail-to-tail. Note that $\boldsymbol{A} \cdot \boldsymbol{B}$ is itself a scalar, which is why the dot product is also called **scalar product**. The dot product is *commutative*,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A} \tag{5}$$

and *distributive*

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}. \tag{6}$$

Geometrically, $\boldsymbol{A} \cdot \boldsymbol{B}$ is the product of $|\boldsymbol{A}|$ times the projection of $\boldsymbol{B}$ along $\boldsymbol{A}$. If the two vectors are parallel then, $\boldsymbol{A} \cdot \boldsymbol{B} = AB$. If they are perpendicular, $\boldsymbol{A} \cdot \boldsymbol{B} = 0$.

4. **Cross product of two vectors.**
The cross product of two vectors is defined by

$$\boldsymbol{A} \times \boldsymbol{B} \equiv AB\sin(\theta)\hat{\boldsymbol{n}} \tag{7}$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing perpendicular to the plane $\boldsymbol{A}$ and $\boldsymbol{B}$. The hat denotes unit vectors. There are two directions perpendicular to any plane: "in" and "out". The ambiguity is resolved by the **right-hand rule**: let your index point in the direction of the first vector and your middle finger in the direction of the second vector, then your thumb indicates the direction of $\hat{\boldsymbol{n}}$ (Fig. 2).

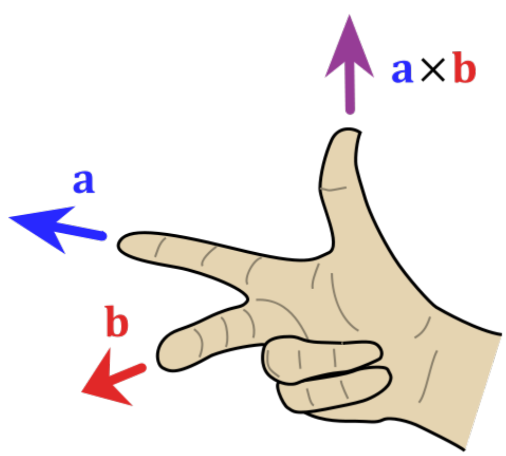

Figure 2: Right-hand rule

$\boldsymbol{A} \times \boldsymbol{B}$ is a vector, which is why the cross product is also called **vector product**. The cross product is *distributive*,

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C}) \tag{8}$$

but not commutative,

$$(\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \times \boldsymbol{B}). \tag{9}$$

Geometrically, $|\boldsymbol{A} \times \boldsymbol{B}|$ is the area of the parallelogram generated by $\boldsymbol{A}$ and $\boldsymbol{B}$. If two vectors are parallel their cross product is zero.

Component form

In the previous section, the vector operations have been described using abstract forms, without any references to a coordinate system. In practice, it is easier to set up Cartesian coordinates x, y, z and work with vector components. Let $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ be unit vectors parallel to the x, y and z axes, respectively. An arbitrary vector $\boldsymbol{A}$ can be expanded in terms of these basis vectors.

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}. \tag{10}$$

The number A_x, A_y and A_z, are the components of $\boldsymbol{A}$; geometrically they are the projections of $\boldsymbol{A}$ along the three coordinate axes. We can reformulate the four vector operations as rules for manipulating components:

1. *To add vectors, add the corresponding components*,

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}} + (A_z + B_z)\hat{\boldsymbol{z}}. \tag{11}$$

2. *To multiply by a scalar, multiply each component individually*,

$$a\boldsymbol{A} = (aA_x)\hat{\boldsymbol{x}} + (aA_y)\hat{\boldsymbol{y}} + (aA_z)\hat{\boldsymbol{z}}. \tag{12}$$

3. *To calculate the dot product, multiply the corresponding components, and sum them up.*

$$\boldsymbol{A} \cdot \boldsymbol{B} = (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}) \cdot (B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}) \tag{13}$$

$$= A_xB_x + A_yB_y + A_zB_z \tag{14}$$

4. *To calculate the cross product, $\boldsymbol{A} \times \boldsymbol{B}$, form the determinant whose first row is $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, whose second row is $\boldsymbol{A}$, and whose third row is $\boldsymbol{B}$.*

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{15}$$

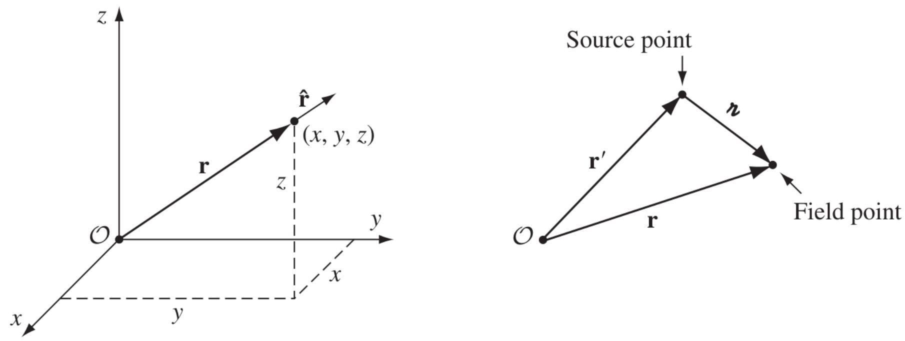

Figure 3: Depiction of position (left) and separation (right) vectors.

Position, Displacement and Separation vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin $(\mathcal{O})$ is called the **position vector** (Fig. 3(Left)).

$$\boldsymbol{r} = x \cdot \hat{\boldsymbol{x}} + y \cdot \hat{\boldsymbol{y}} + z \cdot \hat{\boldsymbol{z}}. \tag{16}$$

The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\boldsymbol{r} = dx \cdot \hat{\boldsymbol{x}} + dy \cdot \hat{\boldsymbol{y}} + dz \cdot \hat{\boldsymbol{z}} \tag{17}$$

In electrodynamics, it is common to encounter problems involving two points, a **source point**, $\boldsymbol{r}'$, where an electric charge is located, and a **field point**, $\boldsymbol{r}$, at which you are calculating the electric or magnetic field (Fig. 3(Right)). We define the **separation vector** from the source point to the field point, with the script letter $\boldsymbol{\mathscr{r}}$.

$$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}' \tag{18}$$

<u>Refresher: Electrostatic</u>

The electrostatic force between two charges is given by

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}} \tag{19}$$

where $\boldsymbol{F}$ is the force on a charge q_1 due to the charge q_2 at a distance r.

Exercise 1 :

1. Derive the dot and cross products for the following couples of vectors:
   (a) $\boldsymbol{A} = (6, 2, 1)$ and $\boldsymbol{B} = (8, 9, 2)$
   (b) $\boldsymbol{A} = (8, 1, 7)$ and $\boldsymbol{B} = (9, 6, 9)$
   (c) $\boldsymbol{A} = (5, 2, 5)$ and $\boldsymbol{B} = (-10, -4, -10)$
   (d) $\boldsymbol{A} = (-3, 8, 2)$ and $\boldsymbol{B} = (0, -8, 1)$
2. Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point $(9, 3, 3)$ to the field point $(6, 1, 7)$. Determine its magnitude, $|\boldsymbol{\mathscr{r}}|$, and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.
3. Find the angle between the body diagonals of a cube.
4. Use the cross product to find the components of the unit vector $\hat{\boldsymbol{n}}$ perpendicular to the shaded plane in Figure 4.

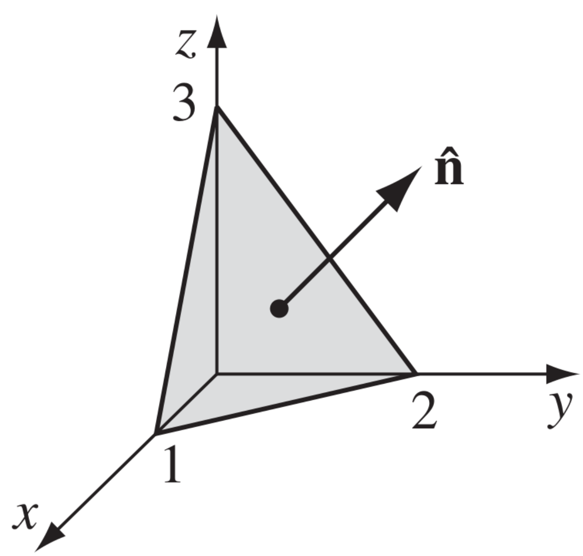

Figure 4: Plane perpendicular to $\hat{\boldsymbol{n}}$.

Solution 1 :

1. According to the definition of dot product among two vectors written in components (13) we have:

   (a) $\boldsymbol{A} \cdot \boldsymbol{B} = (6\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \cdot (8\hat{\boldsymbol{x}} + 9\hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}) = 6 \cdot 8 + 2 \cdot 9 + 1 \cdot 2 = 68$

   (b) $\boldsymbol{A} \cdot \boldsymbol{B} = (8\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} + 7\hat{\boldsymbol{z}}) \cdot (9\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}} + 9\hat{\boldsymbol{z}}) = 8 \cdot 9 + 1 \cdot 6 + 7 \cdot 9 = 141$

   (c) $\boldsymbol{A} \cdot \boldsymbol{B} = (5\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}) \cdot (-10\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} - 10\hat{\boldsymbol{z}}) = 5 \cdot (-10) + 2 \cdot (-4) + 5 \cdot (-10) = -108$

   (d) $\boldsymbol{A} \cdot \boldsymbol{B} = (-3\hat{\boldsymbol{x}} + 8\hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}) \cdot (0\hat{\boldsymbol{x}} - 8\hat{\boldsymbol{y}} + 1\hat{\boldsymbol{z}}) = 3 \cdot 0 + 8 \cdot (-8) + 2 \cdot 1 = -62$

   One can derive the cross product from the definition in Eq. 15:

   (a) $\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\boldsymbol{x}}(A_y B_z - B_y A_z) - \hat{\boldsymbol{y}}(A_x B_z - B_x A_z) + \hat{\boldsymbol{z}}(A_x B_y - B_x A_y) = \hat{\boldsymbol{x}}(2 \cdot 2 - 9 \cdot 1) - \hat{\boldsymbol{y}}(6 \cdot 2 - 8 \cdot 1) + \hat{\boldsymbol{z}}(6 \cdot 9 - 8 \cdot 2) = -5\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} + 38\hat{\boldsymbol{y}}$

   (b) $\boldsymbol{A} \times \boldsymbol{B} = \hat{\boldsymbol{x}}(1 \cdot 9 - 6 \cdot 7) - \hat{\boldsymbol{y}}(8 \cdot 9 - 9 \cdot 7) + \hat{\boldsymbol{z}}(8 \cdot 6 - 9 \cdot 1) = -33\hat{\boldsymbol{x}} - 9\hat{\boldsymbol{y}} + 39\hat{\boldsymbol{y}}$

   (c) $\boldsymbol{A} \times \boldsymbol{B} = \hat{\boldsymbol{x}}(2 \cdot -10 + 4 \cdot 5) - \hat{\boldsymbol{y}}(5 \cdot -10 + 10 \cdot 5) + \hat{\boldsymbol{z}}(5 \cdot -4 + 10 \cdot 5) = 0\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{y}}$

   (d) $\boldsymbol{A} \times \boldsymbol{B} = \hat{\boldsymbol{x}}(8 \cdot 1 + 8 \cdot 2) - \hat{\boldsymbol{y}}(-3 \cdot 1 + 0 \cdot 2) + \hat{\boldsymbol{z}}(-3 \cdot -8 - 0 \cdot -3) = 24\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 24\hat{\boldsymbol{y}}$

2. From the definition of the separation vector (18), we have:

$$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}' = (6 - 9)\hat{\boldsymbol{x}} + (1 - 3)\hat{\boldsymbol{y}} + (7 - 3)\hat{\boldsymbol{z}} = -3\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + 4\hat{\boldsymbol{z}} \tag{20}$$

   The magnitude of a generic vector $\boldsymbol{A} = (A_x, A_y, A_z)$ is given by $|\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$. Therefore, in the case of $\boldsymbol{\mathscr{r}}$ we have:

$$|\boldsymbol{\mathscr{r}}| = \sqrt{(-3)^2 + (-2)^2 + 4^2} = \sqrt{29} \tag{21}$$

   The unit vector associated to $\boldsymbol{\mathscr{r}}$, indicated by $\hat{\boldsymbol{\mathscr{r}}}$, is given by:

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|} = \frac{-3}{\sqrt{29}}\hat{\boldsymbol{x}} + \frac{-2}{\sqrt{29}}\hat{\boldsymbol{y}} + \frac{4}{\sqrt{29}}\hat{\boldsymbol{z}} \tag{22}$$

3. Considering a cube of size ℓ, two vectors representing two possible body diagonals (in total there are 4 possible of them) are given by:

$$\begin{aligned} \boldsymbol{d_1} &= (-\ell, \ell, -\ell) \\ \boldsymbol{d_2} &= (\ell, \ell, -\ell) \end{aligned} \tag{23}$$

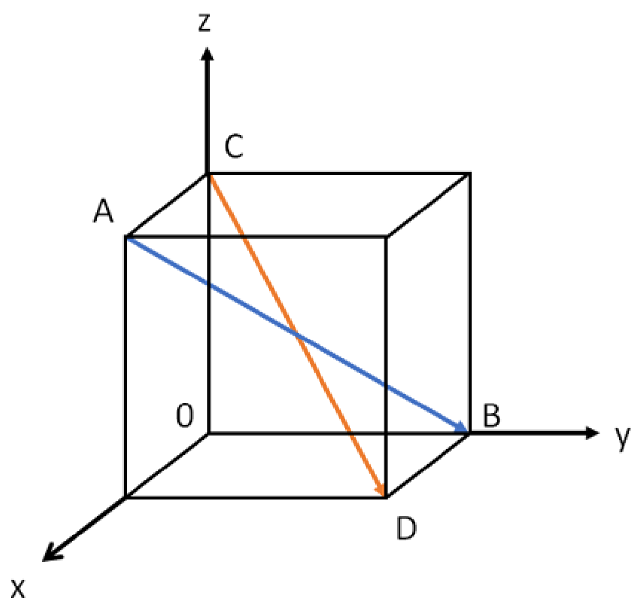

Figure 5: Body diagonals in a cube.

To find the angle θ among the vectors we can employ (4):

$$\boldsymbol{d_1} \cdot \boldsymbol{d_2} = -\ell^2 + \ell^2 + \ell^2 = \ell^2 = |\boldsymbol{d_1}||\boldsymbol{d_2}| \cos\theta, \tag{24}$$

from where we have:

$$\theta = \arccos\left(\frac{\boldsymbol{d_1} \cdot \boldsymbol{d_2}}{|\boldsymbol{d_1}||\boldsymbol{d_2}|}\right) = \arccos\left(\frac{\ell^2}{\sqrt{3}\ell \cdot \sqrt{3}\ell}\right) = \arccos\left(\frac{1}{3}\right) \approx 70.53° \tag{25}$$

4. To find the components of the unit vector $\hat{\boldsymbol{n}}$ perpendicular to the shaded region we compute the cross product among the vectors of two sides of the triangle. From Figure 4 we can see that two possible vectors are:

$$\begin{aligned} \boldsymbol{v}_1 &= -\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \\ \boldsymbol{v}_2 &= -2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}} \end{aligned} \tag{26}$$

Therefore, a vector perpendicular to the shaded region (which is a part of the plane spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$) is:

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = 6\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}} \tag{27}$$

The unit vector $\hat{\boldsymbol{n}}$ is given by:

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|} = \frac{6}{7}\hat{\boldsymbol{x}} + \frac{3}{7}\hat{\boldsymbol{y}} + \frac{2}{7}\hat{\boldsymbol{z}} \tag{28}$$

Exercise 2 :

Two identical water droplets are charged by one extra electron each, such that the electrical force of repulsion compensates for their mutual gravitational force. What is the radius of the droplets ?
Indications: $k_e = 9 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$, $e = 1.6 \times 10^{-19}\,\mathrm{C}$, $G = 6.7 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $\rho_{\text{water}} = 1 \times 10^3\,\mathrm{kg\,m^{-3}}$.

Solution 2 :

The electrical force among the two droplets is given by:

$$F_{\text{el}} = k\frac{e^2}{R^2} \tag{29}$$

where e is the elementary charge and R is the distance among the droplets. The gravitational force, instead, is given by:

$$F_g = G\frac{m^2}{R^2}, \tag{30}$$

where G is the universal constant of gravitation. The mass of each droplets (they are identical) is:

$$m = \frac{4\pi\rho r^3}{3}. \tag{31}$$

Imposing that the two forces are in equilibrium, we obtain the radius of the two droplets:

$$k\frac{e^2}{R^2} = G\frac{m^2}{R^2} \to r^6 = \frac{9}{16\pi^2}\frac{ke^2}{G\rho^2} \tag{32}$$

Exercise 3 :

Three small balls with positive charges $+q_A$ and $+q_B = +q_C$ on the balls A, B and C, respectively, can freely move on a ring. What is the ratio of the charges q_A to q_B, if at the equilibrium the angle from the center of the ring to the charges q_B and q_C is $\pi/3$?

Solution 3 :

Because $q_B = q_c$ the position of the balls B and C are symmetrical relative to the BAC angle bisector line, such that only one of the two charges, B or C, can be considered. The situation is schematized in the following picture.

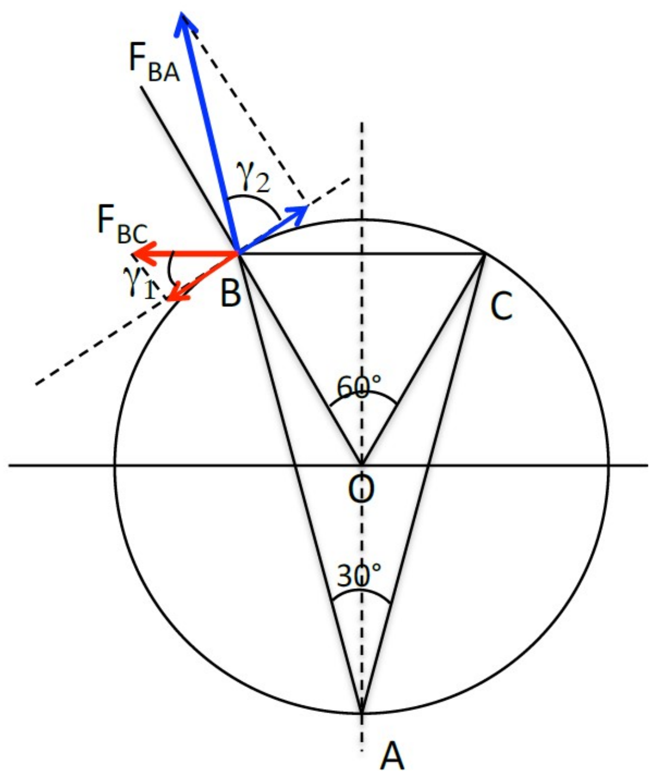

Figure 6: Diagram for the position of the three charges.

Two repulsive forces act on B: one from C (F_{BC}, red arrow in Figure 6) and one from A (F_{AB}, blue arrow in Figure 6) charges. In equilibrium their projections on the tangent to the circle line have to be opposite and equal value.

From the Figure 5 we obtain the following equations:

$$F_{BC}\cos\gamma_1 = F_{BA}\cos\gamma_2 \tag{33}$$

$$k\frac{q_B^2}{BC^2}\cos\gamma_1 = k\frac{q_B q_A}{BA^2}\cos\gamma_2 \tag{34}$$

$$\frac{q_A}{q_B} = \frac{BA^2\cos\gamma_1}{BC^2\cos\gamma_2} \tag{35}$$

Because $OB = OC$ and the angle $BOC = 60°$, $BC = BO$ and the angle $OBC = 60°$. Simple geometrical consideration results in the following:

$$\widehat{\gamma_1} = 90° - \widehat{OBC} = 30° \tag{36}$$

$$\widehat{\gamma_2} = 90° - \widehat{ABO} = 90° - \left(\left(\frac{(180° - 30°)}{2}\right) - 60°\right) = 75° \tag{37}$$

$$BA = \frac{BC}{2\sin 15°} \tag{38}$$

Plugging (38) into (35) we get the ration among q_A and q_B:

$$\frac{q_A}{q_B} = \frac{BC^2 \cdot \cos 30°}{4\sin^2 15° \cdot BC^2\cos 75°} \simeq \frac{0.866}{4 \cdot 0.067 \cdot 0.259} \simeq 12.5 \tag{39}$$

Exercise 4 :

Two charges, $-Q_0$ and $-4Q_0$, are at a distance l apart. These two charges are free to move but do not because there is a third charge nearby. What must be the magnitude of the third charge and its placement in order for the first two to be in equilibrium?

Solution 4 :

The negative charges will repel each other and so the third charge must put an opposite force on each of the original charges. Consideration of the various possible configurations leads to the conclusion that the third charge must be positive and must be between the other two charges. See the following diagram for the definition of variables.

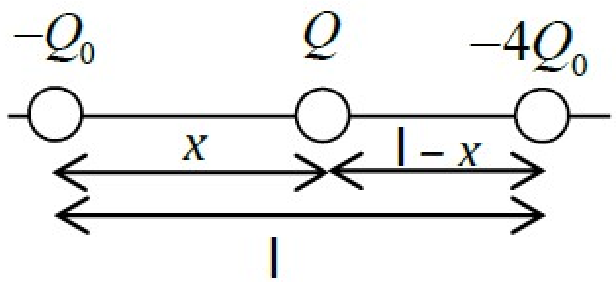

Figure 7: Diagram for the position of the three charges.

For each negative charge, equate the magnitudes of the two forces on the charge. Also note that $0 < x < l$.

For the left charge we have:

$$k\frac{Q_0 Q}{x^2} = k\frac{4Q_0^2}{l^2}, \tag{40}$$

while for the right one:

$$k\frac{4Q_0 Q}{(1-x)^2} = k\frac{4Q_0^2}{l^2}. \tag{41}$$

Combining the previous two relations we obtain:

$$k\frac{Q_0 Q}{x^2} = k\frac{4Q_0 Q}{(1-x)^2} \quad \rightarrow \quad x = \frac{1}{3}l \tag{42}$$

To find the magnitude of the unknown charge we employ (40):

$$Q = 4Q_0\frac{x^2}{l^2} = Q_0\frac{4}{(3)^2} = \frac{4}{9}Q_0 \tag{43}$$

Exercise 5 :

A metal spring, put vertically, has the free length of L_0 (see Figure 8). The length becomes L_g when a ball of mass m is put on top of it. Next, the ball is removed, but two massless point charges $+Q$ and $-Q$ are fixed to the opposite ends of the same spring, which is now in horizontal position. Estimate the value of charge Q, if the spring length becomes L_{el} now? The spring obeys the Hook's law: $F_H = \eta\Delta L$, where ΔL is compression of the spring and η is an unknown constant.

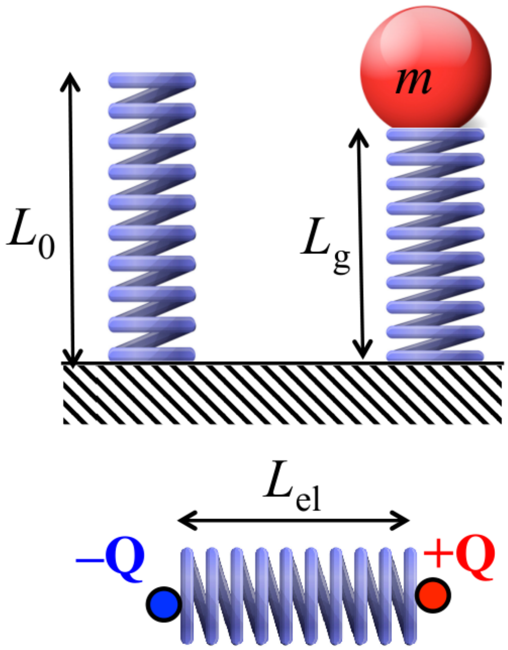

Figure 8: Schematic of the problem in Exercise 5.

Solution 5 :

In vertical position (see Figure 7 on top) the spring experiences two forces, the spring Hook's force and the gravitational force, whose equilibrium is equal to 0:

$$F_H + F_G = -\eta \cdot (L_0 - L_g) + mg = 0, \tag{44}$$

where g is the Earth gravitational force. From the previous relation we obtain the spring constant to be:

$$\eta = \frac{mg}{(L_0 - L_g)} \tag{45}$$

When the two charges are attached (see Figure 7 on bottom), the Hook's force equilibrates Coulomb attraction of the two charges:

$$F_H + F_{el} = -\eta \cdot (L_0 - L_{el}) + k\frac{Q^2}{L_{el}^2} = 0, \tag{46}$$

from where we have:

$$k\frac{Q^2}{L_{el}^2} = \frac{mg \cdot (L_0 - L_{el})}{L_0 - L_g} \Rightarrow Q = L_{el}\sqrt{\frac{mg \cdot (L_0 - L_{el})}{k \cdot (L_0 - L_g)}}. \tag{47}$$