# 1. Electrostatics

1. Charge separation (Triboelectric effect)
2. Electric force = Coulomb force
3. From individual charges to charge distributions
4. Electric field calculations
5. Conservative Coulomb force
6. Electric potential
7. Work done in an electric field
8. Gauss's law
9. Application of Gauss's law
10. Electrostatic potential in an ion crystal
11. Conductor with charges in the stationary state
12. Force on charges in a conductor sheet
13. Poisson and Laplace equations

# 1.13 Poisson and Laplace equations

Comment: When the charge distribution is not known the previous formalisms for calculating electrical fields are of limited use.
But if electrical potential values on specific surfaces in space are known,
the relevant electrical field can still be derived (uniqueness theorem).

After this section
you will know a formalism which does not require the knowledge of the charge density.
The formalism is based on the Laplace equation.

# 1.13 Poisson and Laplace equations

So far: $\vec{E} = -\vec{\nabla}\phi(\vec{r})$

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla}^2\phi(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

**Poisson equation** $$\boxed{\vec{\nabla}^2\phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}}$$

"del squared"

$$\vec{\nabla}^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

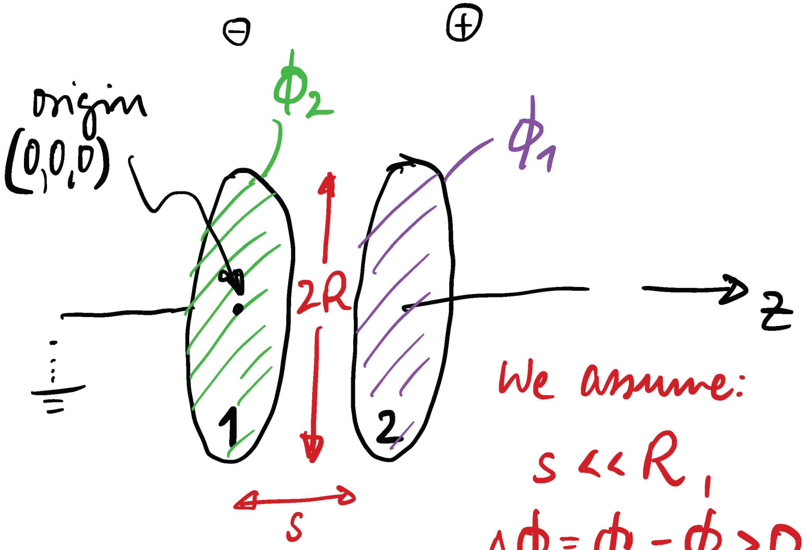
**= the Laplacian**

For $\rho = 0$ (space without charges):

**Laplace equation** $$\boxed{\vec{\nabla}^2\phi(\vec{r}) = 0}$$

"second-order partial differential equation"

# Laplace equation applied and solved for a disk-shaped plate capacitor




We assume:
$s \ll R$,
$\Delta\phi = \phi_1 - \phi_2 > 0$

1) Symmetry analysis:
with respect to z-axis
cylindrical symmetry

2) We want to find
$\phi(x,y,z)$ or better
$\phi(r,\theta,z)$ between the two plates. There: $\rho = 0$

3) Hence, we exploit the Laplace equation to find $\phi$:

$$\vec{\nabla}^2 \phi = 0$$

4) Due to symmetry analysis we use $\vec{\nabla}^2$ in cylindrical coordinates:

$$\vec{\nabla}^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

(Math Toolbox I)

5) We consider a point r inside the capacitor far away from the edges (in the center). If $r \ll R$, one can assume a uniform field:
$\Rightarrow \partial\phi/\partial\theta = 0$ and $\partial\phi/\partial r \approx 0$

6) Final Laplace Eq.:

$$\frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } r \ll R$$

7) General solution (integrating twice both sides):

$$\phi(z) = c_1 \cdot z + c_2$$

8) Find $c_1$ & $c_2$ from boundary conditions:

$\phi(0) = \phi_2 = c_2$ and

$\phi(z=s) = \phi_2 + \Delta\phi = c_1 s + \phi_2$

$\Rightarrow c_1 = \frac{\Delta\phi}{s}$

9) Solution between plates for $r \ll R$:

$$\phi(z) = \frac{\Delta\phi}{s} \cdot z + \phi_2$$

"linearly increasing $\phi$"

10) $\vec{E} = -\text{grad}\,\phi = -\frac{\Delta\phi}{s}\hat{z}$

Added note:

There was a question how to solve the Laplace equation if the solution depends on two coordinates instead of one.

There are two aspects to consider:

Partial solutions of the problem can be found by considering one-dimensional Laplace equations (in separate directions).
These solutions are valid also for the higher dimensional Laplace equation but do not represent the general solution.

For the general solution one follows an approach for which one considers that the general formula of the total potential function is composed of a product of two one-dimensional functions e.g. X(x) and Y(y) (separation of variables). One puts this product ansatz into the Laplace equation, to get a "new" Laplace equation . By this equation one separates terms for X(x) and Y(y). One of the terms is set to a positive constant and one to the negative number such that their sum always fulfills the "new" Laplace equation. Now the two directions can be solved separately and are connected via the auxiliary constant. Both equations are second-order partial differential equations.
Note that at this point one often needs to guess the appropriate harmonic function(s), insert them and test if they can fulfill the two equations at the same time plus the boundary conditions (potential values on the boundaries.

The Laplace equations in 2D and 3D dimensions do not have a simple analytical solution in general and will not be considered further in PHYS201(d).

# Practical implications with Laplace equation

- Analyze the symmetry of the electrostatic problem (e.g. potential between two charged planes, between two cylindrical charge distributions)

- **Choose the most appropriate coordinate system** (cartesian, cylindrical, spherical)

- Search for the del squared operator applied to a scalar function in the relevant coordinate system (Math Tool Box I), and **consider only its derivatives relevant for the problem**

- Integrate twice considering indefinite integrals by **introducing constants of integration** (**exercise on problem sheet 4**).

- The final function needs to be checked against the boundary conditions (**potential values at specific points in space**); **putting-in these "boundary" values** determines the values of the integration constants

- Once a function is found (without knowing the charges beforehand!) and **it fulfills the Laplace equation (check!),** then the only unique solution has indeed been found (**theorem of uniqueness**, see next page).

# Uniqueness theorem, harmonic functions

<u>**Solutions of the Laplace equation**</u> have the following characteristics:

1) **Theorem of uniqueness**
   Consider a volume $\Omega$ (single- or multiple-connected) and its surface *S*:
   If the potential $\phi$ is given everywhere on *S (boundary condition)*,
   this solution $\phi(\vec{r})$ is **the only one** for the volume $\Omega$

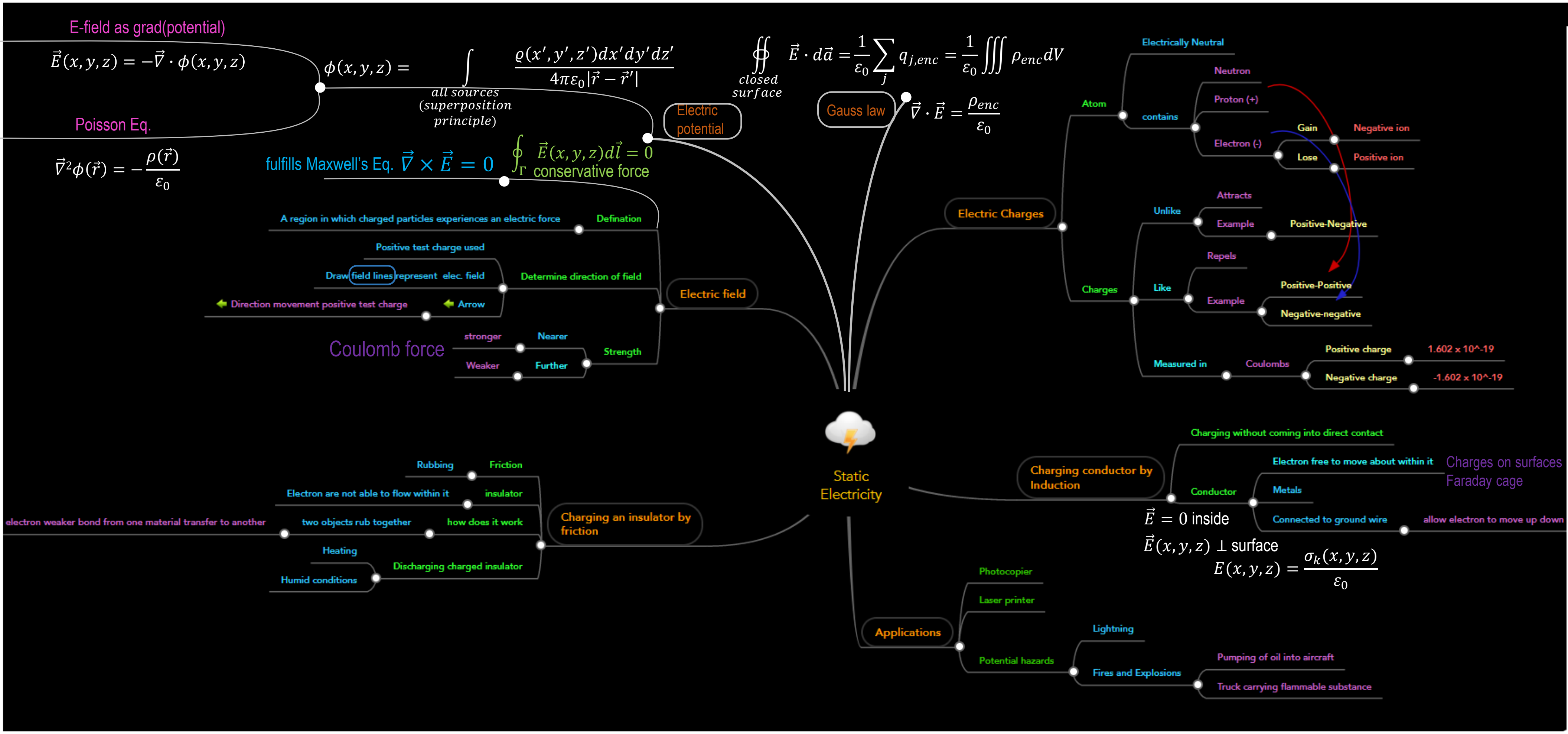
2) Consider point P(*x,y,z*) being part of $\Omega$, and
$\Sigma$ denotes a sphere of radius *R* around the point P, then $\phi(\mathrm{P}) = \dfrac{1}{4\pi R^2}\oint_\Sigma \phi(\vec{r})\, dS$

This is the average value of $\phi(\vec{r})$ on *S*.

3) The class of functions that solve the Laplace equation are called **harmonic functions** (twice continuously differentiable functions)

# Mindmap I for Electrostatics

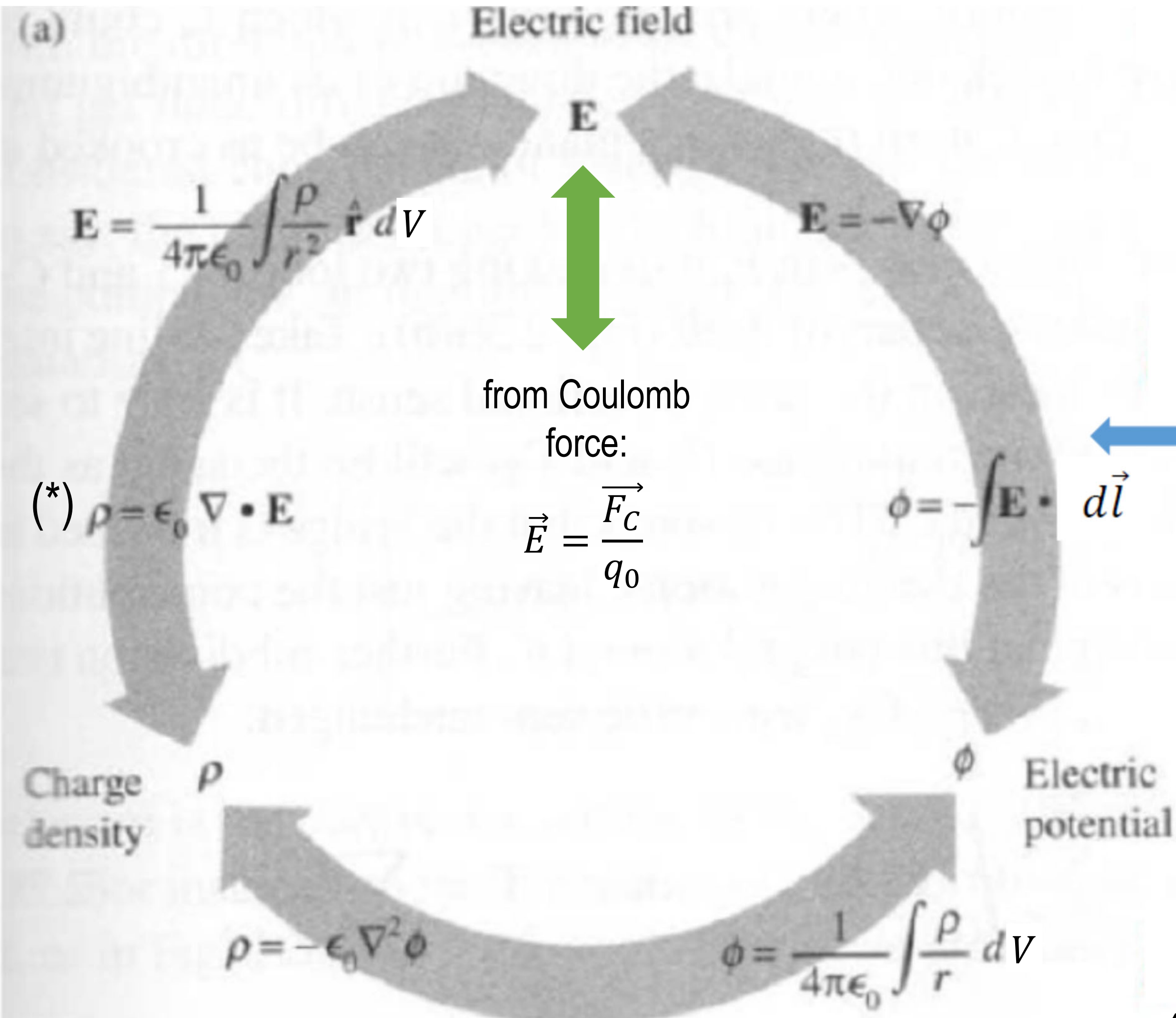
Further completed the mindmap




Adapted from: https://www.mindmeister.com/79285653/static-electricity?fullscreen=1#

EPFL

# Summary of Chapter 1 (inspired by Purcell)




taken from Purcell

Closed line integral for static charge distribution:

$$\oint_\Gamma \vec{E}\, d\vec{l} = 0 \qquad \vec{\nabla} \times \vec{E} = 0$$

Conservative force

Electrostatic potential energy $U_p = q\phi(\vec{r})$

(*) Gauss's law

$$Flux\ \Phi = \underset{\substack{closed\\surface\\integral}}{\oiint} \vec{E}\, d\vec{a} = \frac{1}{\varepsilon_0}\sum_i q_i = \frac{1}{\varepsilon_0}\iiint \rho\, dV$$

# Recap: Different usage of del-operator

del-operator: In Cartesian coordinates, $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \nabla \cdot \mathbf{F}$$

Scalar product used in Gauss' law (divergence):

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\operatorname{curl} \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \nabla \times \mathbf{A}$$

Vector product used in Maxwell equation (curl):

$$\vec{\nabla} \times \vec{E} = 0$$

$$\operatorname{grad} \phi = \hat{\mathbf{x}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \phi}{\partial z} = \nabla \phi$$

"Multiplication" used with potential function (gradient):

$$\vec{E}(x, y, z) = -\vec{\nabla} \cdot \phi(x, y, z)$$

(formulas taken from MathToolbox I)

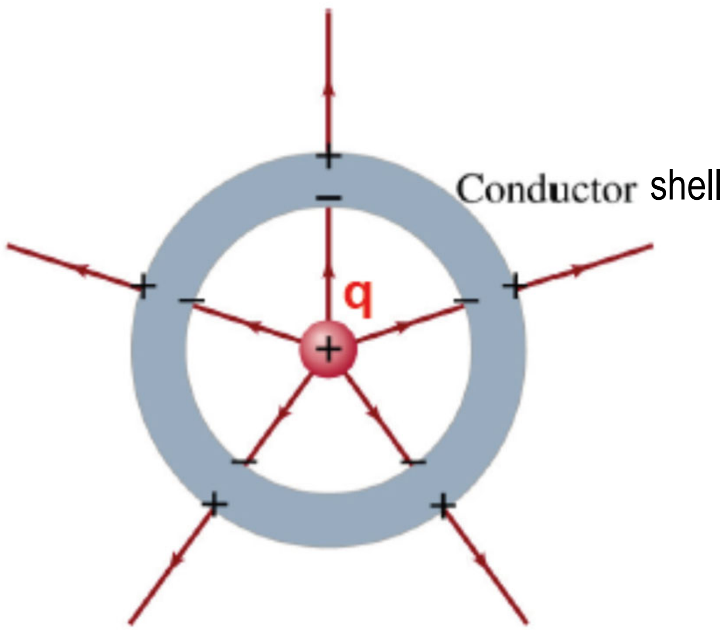
## How much charge is distributed on the inner surface of the *neutral* conductor shell (spherical shell around charge *q*)?

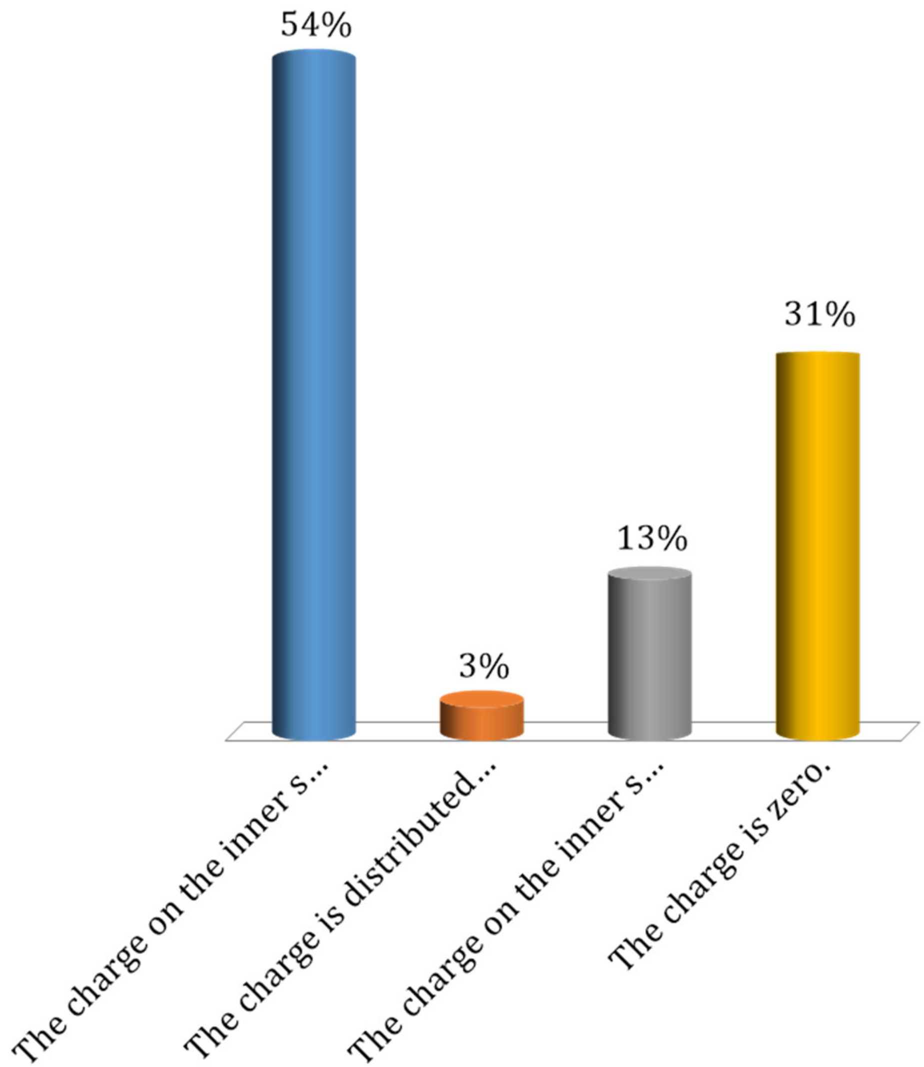
A. The charge on the inner shell amounts to –q.

B. The charge is distributed over a large area and its absolute value is larger than q.

C. The charge on the inner surface is negative and its absolute value is smaller than q.

D. The charge is zero.




Solution: imagine a Gauss's surface which is just following the central position inside the conductor shell and which surrounds the charge q plus the unknown charge distributed on the inner surface of the shell.
As the electric field is zero in the conductor, and Gauss's law must hold, the total charge enclosed must add up to zero. This fixes the charges on the inner surface to -q. in total spread of the surface.