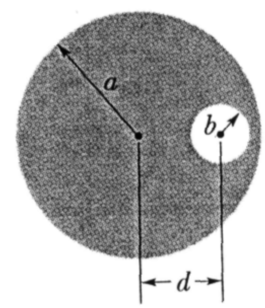
Exercise sheet 9: Induction law, generator, power

13/11/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Cavity in a wire/Category II/After training: 30 min)

(I) What is the magnetic field $\vec{B}$ inside a cylindrical conducting wire carrying a uniform current density j, at a distance r away from the axis? Use Ampere's law.
(II) The figure shows the section of a long cylindrical conducting material of radius a, within which is a cylindrical cavity of radius b. The cylinders have parallel axes separated by a distance d, see the figure. A current i is uniformly distributed within the conducting material (in dark grey in the figure). Answer the following questions. Hint: Use the result of the preliminary question (I) and the superposition principle.



a) Calculate the magnetic field along the axis of the cavity.

b) Show that the magnetic field is constant in the cavity.

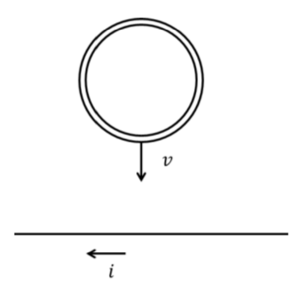
Exercise 2.
(Infinite Solenoid/Category II)
Consider an infinitely long solenoid with n-turns per unit length. The long axis is along the $+\vec{z}-$direction. The current I flows along the $\vec{e}_\varphi$ direction in cylindrical coordinates. We assume the solution developed in the lecture concerning the uniform magnetic field $\vec{B}$ inside the solenoid:

$$\vec{B} = \mu_0 n I \vec{e}_z \tag{1}$$

Use the formula 1, the definition for magnetic flux $\Phi_m = \iint_{\text{area enclosed by a path}\,\Gamma} \vec{B} \cdot d\vec{a}$ to show that $\vec{A} = \frac{1}{2}\mu_0 n r I \vec{e}_\varphi$ inside the solenoid and $\vec{A} = \frac{1}{2}\mu_0 n \frac{b^2}{r} I \vec{e}_\varphi$ outside the solenoid. Here, $\vec{A}$ is the vector potential, b is the radius of ultrathin solenoid and r is the distance from the long central axis. Considering the solution for the vector field outside the solenoid: Is there a magnetic field outside the solenoid?

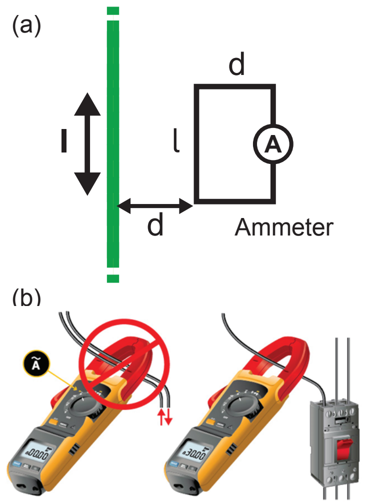
Exercise 3.
(Ring falling towards a wire / Category I)

Consider a conducting ring falling towards a conducting wire carrying a current i, see figure. According to the induction law, a current i_{ind} will be induced in the ring itself. In what direction?



Exercise 4.
(Current clamp / Category I)

Imagine a straight electric power cable (green line in sketch (a) which is plugged into the electric grid ($I = I_0 \sin \omega t$ with $\omega = 2\pi f$). The frequency f in the electric grid amounts to 50 Hz. You want to find out the current flowing in the cable without removing the isolation. One possibility is to put a wire loop next to the cable and detect the inducted current.



a) Calculate the electromotive force emf in a rectangular wire loop with length $l = \frac{1}{\ln(2)}$ m and width d, which is placed next to the cable at distance d as indicated in sketch (a).

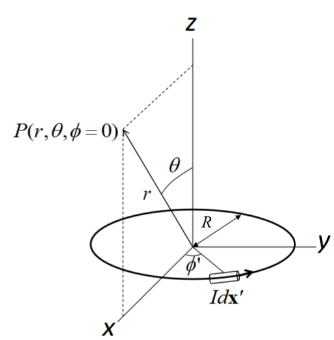
b) The wire has a resistance of $R = 1\ \Omega$. In the wire loop, a peak current of $I_{peak} = 1$ mA is measured with an ammeter. Calculate the current flowing inside the power cable.

c) The same working principle is also used in commercial devices. Sketch (b) shows an excerpt from a manual of a current clamp. Can you explain the instructions? Suppose the device works exactly as in the previous question, and that whatever inside the clamp plays the role of the green wire.

d) Does a current clamp based on induction also work for a DC current?

Exercise 5.
(Vector potential and magnetic field for a circular current loop/Advanced Category III)

Consider the problem of a circular loop of radius R, lying in the x, y-plane, centered at the origin, and carrying a current I, as shown in the figure.



a) Calculate the vector potential $\vec{A}(r, \theta, \phi)$ for $r \gg R$ at point P in the x, z-plane ($\phi = 0$) as sketched. What is the direction of $\vec{A}$ following your calculation. Can you explain the orientation with symmetry arguments.
Hint: $\left(r^2 + R^2 - 2rR\sin\theta\cos\phi'\right)^{-1/2} \cong \frac{1}{\sqrt{r^2+R^2}}\left(1 + \frac{rR}{r^2+R^2}\sin\theta\cos\phi'\right)$

b) How does $\vec{A}$ vary as a function of ϕ at fixed r and θ? For this consider the direction of $\vec{A}$ when P is either in the x, z-plane ($\phi = 0$) as sketched or assumed to be in the y, z-plane ($\phi = \pi/2$). Does this variation as a function of ϕ agree with the symmetry arguments?

c) Show that the result of part (a) can be expressed as $\vec{A}(r, \theta, \phi) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$ where, $\vec{m} = I\vec{a}$ is the *magnetic dipole moment* of the current loop and the vector $\vec{a}$ is the directed area of the region surrounded by the loop.

d) Calculate the magnetic field $\vec{B}(r, \theta, \phi)$ for $r \gg R$ from $\vec{A}(r, \theta, \phi)$.

Solution 1.

1. Using Ampere's law (Fig. 1):

   $\oint_\Sigma \vec{B} \cdot d\vec{s} = \mu_0 I \Rightarrow B2\pi r = \mu_0 j\pi r^2 \Rightarrow \vec{B} = \frac{\mu_0 jr}{2}\hat{\theta}$. Note that the current is assumed to be along the z-axis pointing out of the plane of the paper. Hence the unit vector $\hat{\theta}$ representing cylindrical coordinates points to as indicated in Figure 1 (a). The magnetic field on the x-axis inside and outside the wire is plotted in Figure 1 (b).

   Figure 1 (a) shows the used symbols.

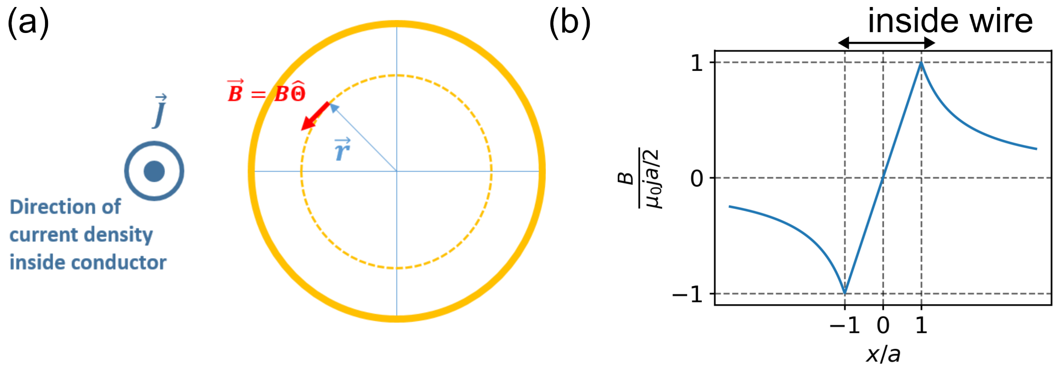


Figure 1: (a) Cylindrical conductor carrying uniform current density. Sketch for calculating the magnetic field inside the conductor assuming the specified direction for the current density. The circumference that is defined by the radius vector $\vec{r}$ is $\Sigma = 2\pi r$. An infinitesimal part of Σ is represented by ds. The vector $d\vec{s}$ is directed tangentially to the circumference Σ. (b) Magnetic field of the cylindrical conducting wire plotted in a cartesian coordinate system on the x-axis. The field is zero in the center.

2. a. Total field is given by the superposition of two fields generated by two cylinders: one carrying $\vec{j}$ and the other one (the cavity) $-\vec{j}$. The current density is $j = \frac{i}{\pi(a^2-b^2)}$. The first wire generates a field inside the cavity along its axis $\vec{B}(d) = \hat{\theta}\frac{\mu_0 jd}{2} = \hat{\theta}\frac{\mu_0 id}{2\pi(a^2-b^2)}$. The second wire assumed to exist in the cavity does not generate any field in its own center (Figure 1 (b)). To obtain a general expression we consider Fig 2.

   b. The reference system and all geometrical quantities to solve this step are shown in Fig. 2. Let us compute the magnetic field inside the cavity at an arbitrary point $\vec{P}$. By superposition principle we sum the contribution of the two wires $\vec{B}(\vec{P}) = \vec{B}_1(\vec{P}) + \vec{B}_2(\vec{P}) = \frac{\mu_0 jr_1}{2}\hat{\theta}_1 + \frac{\mu_0 jr_2}{2}(-\hat{\theta}_2)$. We express the unit vectors of the two magnetic fields in terms of cartesian unit vectors $\hat{x}$ and $\hat{y}$ (Fig. 2):

$$\hat{\theta}_1 = -\sin(\alpha)\hat{x} + \cos(\alpha)\hat{y}$$
$$-\hat{\theta}_2 = \sin(\beta)\hat{x} - \cos(\beta)\hat{y}$$

   Expressing cosine and sine terms as a function of the distance yields

$$\hat{\theta}_1 = -\frac{y}{r_1}\hat{x} + \frac{x+d}{r_1}\hat{y}$$
$$-\hat{\theta}_2 = \frac{y}{r_2}\hat{x} - \frac{x}{r_2}\hat{y}$$

   Exploiting these calculations we find $\vec{B}(\vec{P}) = \frac{\mu_0 jr_1}{2}\left(-\frac{y}{r_1}\hat{x} + \frac{x+d}{r_1}\hat{y}\right) + \frac{\mu_0 jr_2}{2}\left(\frac{y}{r_2}\hat{x} - \frac{x}{r_2}\hat{y}\right) = \frac{\mu_0 jd}{2}\hat{y}$.

   We thus conclude that the total field inside the cavity $\vec{B}(\vec{P}) = \frac{\mu_0 jd}{2}\hat{}$ does not depend on spatial coordinates. Its magnitude is proportional to the current density and the separation d between the centers of the two cylinders.

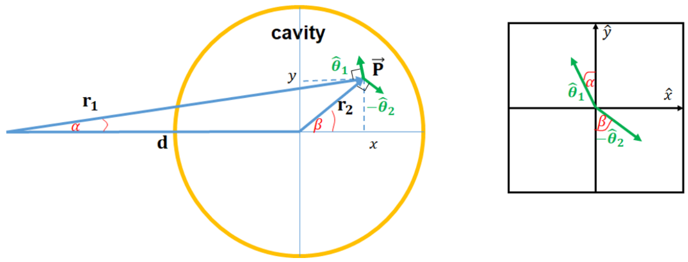


Figure 2: Cavity is drawn with geometrical parameters and reference system.

Solution 2.

The schematic of the exercise is shown in figure 3.

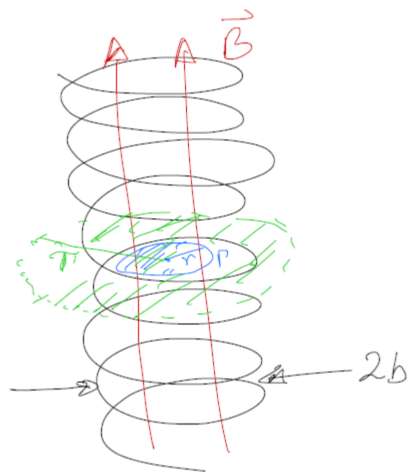


Figure 3: Schematic of an infinitely long Solenoid of radius b, with areas inside (blue) and outside (green) marked.

$$\vec{B} = \mu_0 n I \vec{e}_z \tag{2}$$

Inside the coil ($r < b$):

$$\begin{aligned}\Phi_m &= \iint_{\text{blue surface}} \vec{B} \cdot d\vec{a} \\ &= |\vec{B}| \cdot \pi r^2 \\ &= \mu_0 n I \pi r^2\end{aligned} \tag{3}$$

Using Stoke's theorem,

$$\iint \vec{B} \cdot d\vec{a} = \iint \vec{\nabla} \times \vec{A} \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l} \tag{4}$$

Symmetry analysis of $\vec{A}$ suggests that it must be a vector field circulating around the vector field $\vec{B}$ in a plane perpendicular to $\vec{B}$, as $\vec{B}$ is assumed to consist of infinitely long, straight field lines: $\vec{A} = A_\varphi \vec{e}_\varphi$. Using cylindrical coordinates,

$$\Rightarrow \oint \vec{A} \cdot d\vec{s} = A_\varphi \cdot 2\pi r \tag{5}$$

Use equation 3 to get $\Phi_m = \mu_0 n I \pi r^2 = A_\varphi \cdot 2\pi r$. $\Rightarrow A_\varphi = \frac{1}{2}\mu_0 n I r \vec{e}_\varphi$

Outside the coil ($r > b$):

$$\begin{aligned}\Phi_m &= \iint_{\text{green surface}} \vec{B} \cdot d\vec{a} \\ &= |\vec{B}| \cdot \pi b^2 \\ &= \mu_0 n I \pi b^2\end{aligned} \tag{6}$$

Equation 5 still reads: $\oint \vec{A} \cdot d\vec{s} = A_\varphi \cdot 2\pi r$.

$$\Rightarrow \vec{A}_\varphi = \frac{1}{2}\mu_0 n I \frac{b^2}{r}\vec{e}_\varphi \tag{7}$$

Now for $\vec{B}$ for $r > b$: Equation 7 needs to be considered in $\vec{B} = \vec{\nabla} \times \vec{A}$. Remember that $A_z = 0$ and $A_r = 0$, only $A_\varphi \neq 0$.

$$\vec{\nabla} \times \vec{A} = (\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z})\vec{e}_r + (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\vec{e}_\varphi + \frac{1}{r}(A_\varphi + r\frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi})\vec{e}_z \tag{8}$$

The only non-zero terms here are: $\frac{A_\varphi}{r}$ and $\frac{1}{r}r\frac{\partial A_\varphi}{\partial r}$.

$$\frac{A_\varphi}{r} = \frac{1}{2}\mu_0 n I \frac{b^2}{r^2} \tag{9}$$

$$\frac{1}{r}r\frac{\partial A_\varphi}{\partial r} = \frac{\partial A_\varphi}{\partial r} = \frac{\partial}{\partial r}(\frac{1}{2}\mu_0 n I \frac{b^2}{r}) = -\frac{1}{2}\mu_0 n I \frac{b^2}{r^2} \tag{10}$$

Using 9 and 10 in 8, $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{A_\varphi}{r} + \frac{1}{r}r\frac{\partial A_\varphi}{\partial r} = \frac{1}{2}\mu_0 n I \frac{b^2}{r^2} + (-\frac{1}{2}\mu_0 n I \frac{b^2}{r^2}) = 0$ The vector field $\vec{A}$ calculated before is consistent with our original assumption that $B = 0$ outside of the infinitely long solenoid.

Solution 3.

The ring is falling. Using Ampere's law for a magnetic field generated by a wire $B \propto r^{-1}$ we know that the magnetic flux in the ring is increasing. Geometry of the system is illustrated in Fig. 4. To counteract this increasing magnetic

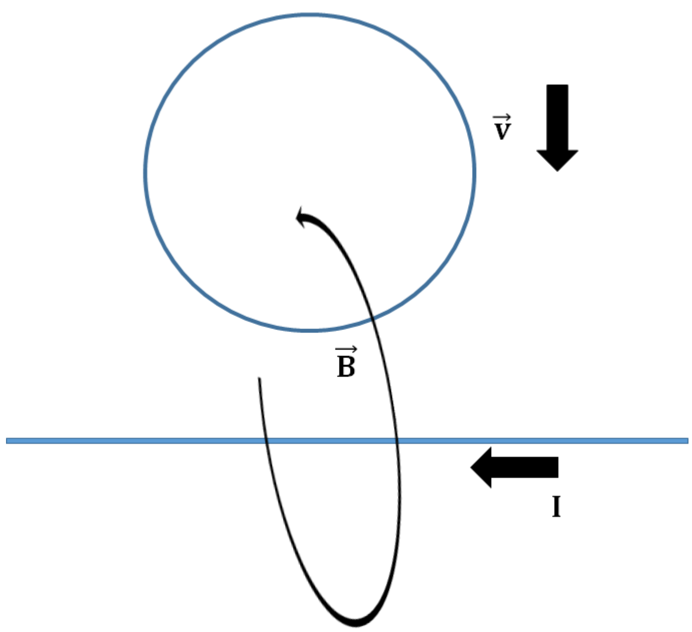


Figure 4: Sketch to understand the geometry of system when the ring is falling.

flux the current will start flowing counterclockwise in the ring.

Solution 4.

a) emf is $\Delta V = -\frac{d\Phi}{dt}$.
We must find the flux
$\Phi = \int_\Sigma \vec{B}\cdot d\vec{\sigma} = \frac{\mu_0 I(t) l}{2\pi}\int_d^{2d}\frac{dr}{r} = \frac{\mu_0 I(t)}{2\pi} l\cdot ln(2)$.
We conclude that
$\Delta V(t) = -\frac{d}{dt}\left(\frac{\mu_0 I(t)}{2\pi} l\cdot ln(2)\right) = -I_0\omega\cos(\omega t)\frac{\mu_0 l}{2\pi} ln(2) = -I_0\omega\cos(\omega t)\frac{\mu_0}{2\pi}$.

b) $|\Delta V_{peak}| = RI_{peak} = I_0\omega\frac{\mu_0 l}{2\pi} ln(2) \Rightarrow I_0 = RI_{peak}\frac{2\pi}{\mu_0\omega l\cdot ln(2)} = \frac{1000}{20\pi}\text{A} = 15.9$ A.

c) The current flows in opposite direction in the two wires. Measuring at the same time these two conductors does not detect any current because for opposite current the generated magnetic fields cancel each other out giving zero total field.

d) For a DC current the field is constant in time therefore the time derivative of the flux, i.e. the electromotive force, is zero. This method does not work for DC measurements.

Solution 5.

a) The vector potential of the loop is given by $\vec{A}(\vec{r}) = \dfrac{\mu_0 I}{4\pi}\oint\dfrac{d\vec{l'}}{|\vec{r}-\vec{r'}|}$. Considering the geometry of the problem we have: $\vec{r} = r\hat{e}_r(\theta,\phi=0)$ and $\vec{r'} = R\hat{e}_{r'}(\theta'=\frac{\pi}{2},\phi')$. From the radial unit vector in spherical coordinate $\hat{e}_r(\theta,\phi) = \sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}+\cos\theta\hat{z}$, we have $\vec{r} = r(\sin\theta\hat{x}+\cos\theta\hat{z})$ and $\vec{r'} = R(\cos\phi'\hat{x}+\sin\phi'\hat{y})$. Hence, $|\vec{r}-\vec{r'}| = \left(r^2+R^2-2rR\sin\theta\cos\phi'\right)^{1/2}$. The loop element vector is $d\vec{l'} = R\,d\phi'\hat{e}_{\phi'} = R\,d\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})$ where we express the azimuthal unit vector in Cartesian coordinates. Therefore, we need to evaluate the following integration to obtain $\vec{A}$:

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\,d\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})}{\left(r^2+R^2-2rR\sin\theta\cos\phi'\right)^{1/2}}.$$

Using the hint given in the exercise, we obtain

$$\vec{A} = \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{r^2+R^2}}\int_0^{2\pi}d\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})\left(1+\frac{rR}{r^2+R^2}\sin\theta\cos\phi'\right).$$

The x component of vector potential, A_x vanishes as follows

$$\begin{aligned}A_x &= \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{r^2+R^2}}\int_0^{2\pi}d\phi'\left(-\sin\phi'-\frac{rR}{r^2+R^2}\sin\theta\sin\phi'\cos\phi'\right)\\ &= \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{r^2+R^2}}\left[\cos\phi'+\frac{rR}{r^2+R^2}\sin\theta\left(\frac{1}{2}\left(\frac{1}{2}\cos 2\phi'\right)\right)\right]_0^{2\pi} = 0.\end{aligned}$$

The non-vanishing y component of vector potential, A_y is

$$\begin{aligned}A_y &= \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{r^2+R^2}}\int_0^{2\pi}d\phi'\left(\cos\phi'+\frac{rR}{r^2+R^2}\sin\theta\cos^2\phi'\right)\\ &= \frac{\mu_0 I}{4\pi}\frac{R}{\sqrt{r^2+R^2}}\left[\sin\phi'+\frac{rR}{r^2+R^2}\sin\theta\left(\frac{1}{2}\left(\phi'+\frac{1}{2}\sin 2\phi'\right)\right)\right]_0^{2\pi}\\ &= \frac{\mu_0 I}{4\pi}\left(\frac{r\pi R^2}{\left(r^2+R^2\right)^{3/2}}\right)\sin\theta.\end{aligned}$$

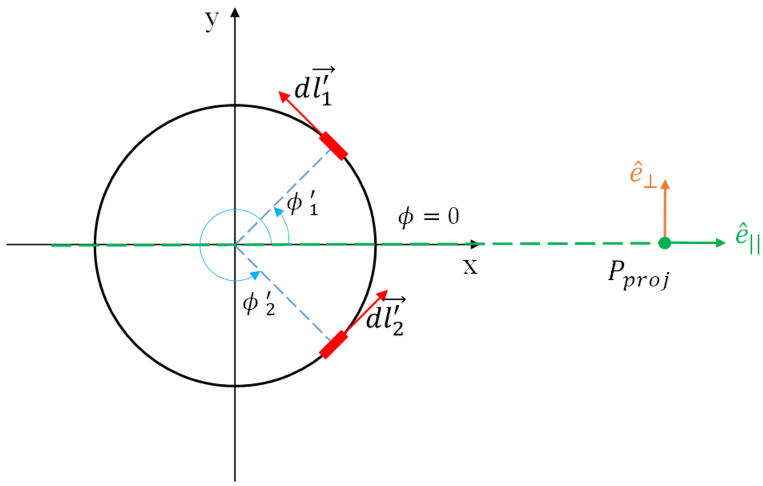


Figure 5: Two mirror image elements on the loop with respect to axis $\phi = 0$.

Contribution of each element $d\vec{l'}$ to the vector potential is $d\vec{A} \propto \dfrac{d\vec{l'}}{|\vec{r}-\vec{r'}|}$. From this we see that $d\vec{A}$ is parallel to $d\vec{l'}$. Hence, we now analyze the symmetries of elements $d\vec{l'}$ around the loop to describe their respective contribution of $d\vec{A}$ to the total vector potential. Consider two elements $d\vec{l_1'}$ and $d\vec{l_2'}$ which are mirror images with respect to the axis $\phi = 0$ (which is defined by projecting the observation point P onto the $x-y$ plane). By decomposing $d\vec{A}$ to its parallel and perpendicular components with respect to the axis $\phi = 0$, we have: $d\vec{A} = d\vec{A}_{\|} + d\vec{A}_{\perp}$. It is evident from Fig. 5 that $d\vec{A}_{\|,1} = -d\vec{A}_{\|,2}$ and $d\vec{A}_{\perp,1} = d\vec{A}_{\perp,2}$. Integrating over full loop results in a non-vanishing $\vec{A}_{\perp}$ while $\vec{A}_{\|}$ vanishes.

b) The same arguments from part (a) holds for any ϕ as it is shown in Fig. 6. Therefore, for each ϕ we will have a non-vanishing $\vec{A} = \vec{A}_{\perp} = A_\phi \hat{e}_{\phi'}$ where $A_\phi = \dfrac{\mu_0 I}{4\pi}\left(\dfrac{r\pi R^2}{(r^2+R^2)^{3/2}}\right)\sin\theta$.

c) For the case of $r \gg R$ we can approximate the result of the previous part as $\vec{A} = \vec{A}_{\perp} = A_\phi \hat{e}_\phi = \dfrac{\mu_0 I}{4\pi}\left(\dfrac{r\pi R^2}{r^3}\right)\sin\theta \hat{e}_\phi$. Knowing that $\vec{m} = I\vec{a} = I\pi R^2 \hat{z}$ and $\vec{r} = r\hat{e_r}$ we have

$$
\begin{aligned}
\vec{m}\times\vec{r} = Ir\pi R^2 &\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} \\
&= Ir\pi R^2\left(\hat{x}(-\sin\theta\cos\phi) + \hat{y}(\sin\theta\sin\phi)\right) \\
&= Ir\pi R^2 \sin\theta \hat{e}_\phi
\end{aligned}
$$

Hence, $\vec{A}(r,\theta,\phi) = \dfrac{\mu_0}{4\pi}\dfrac{\vec{m}\times\vec{r}}{r^3} = \dfrac{\mu_0}{4\pi}\dfrac{\vec{m}\times\hat{r}}{r^2}$.

d) The magnetic field is given by $\vec{B} = \vec{\nabla}\times\vec{A}$. In spherical coordinates by considering $A_r = 0$ and $A_\theta = 0$ we have:

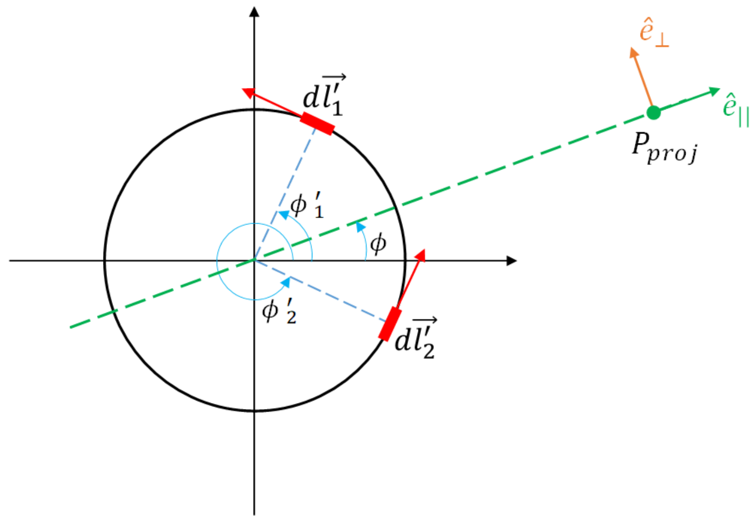


Figure 6: Two mirror image elements on the loop with respect to axis ϕ.

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left[\hat{e}_r \left(\frac{\partial}{\partial \theta} (r\sin\theta A_\phi) \right) + r\hat{e}_\theta \left(-\frac{\partial}{\partial r} (r\sin\theta A_\phi) \right) \right].$$

Using $A_\phi = \frac{\mu_0 I}{4\pi} \left(\frac{\pi R^2}{r^2} \right) \sin\theta$, we obtain $\vec{B} = B_r\hat{e}_r + B_\theta\hat{e}_\theta$ with

$$B_r = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(r\sin\theta \times \frac{\mu_0 I}{4\pi} \left(\frac{\pi R^2}{r^2} \right) \sin\theta \right)$$
$$= \frac{2\mu_0}{4\pi} \frac{I\pi R^2}{r^3} \cos\theta = \frac{2\mu_0 m}{4\pi r^3} \cos\theta,$$

$$B_\theta = \frac{-1}{r\sin\theta} \frac{\partial}{\partial r} \left(r\sin\theta \times \frac{\mu_0 I}{4\pi} \left(\frac{\pi R^2}{r^2} \right) \sin\theta \right)$$
$$= \frac{\mu_0}{4\pi} \frac{I\pi R^2}{r^3} \sin\theta = \frac{\mu_0 m}{4\pi r^3} \sin\theta.$$

This result is consistent with the formulas given in the lecture ("magnetic field of a magnetic dipole / current loop").