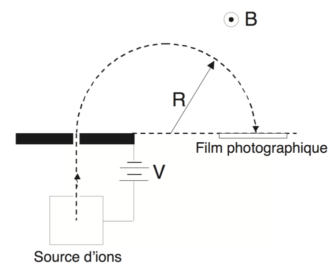
Exercise sheet 8: Magnetic fields, Ampère-Laplace (Biot-Savart) law, Ampère's law

6/11/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Thomson's discovery/From Griffiths/Category I)
In 1897, J.J. Thomson"discovered" the electron by measuring the charge-to-mass ratio of "cathode rays". He selected particles of a specific speed via the following"trick": He passed the beam through uniform crossed electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively. They were mutually perpendicular and both of them were perpendicular to the beam. He adjusted the magnitude of the electric field E until he got zero deflection for a fixed magnetic field B_0. What was the speed of the particles in terms of E and B_0 which had a straight trajectory?

Exercise 2.
(Mass Spectrometer/Category I)



We consider the mass spectrometer represented in the sketch. Two isotopes of lithium, with an atomic masses of 6 au and 7 au, are ionized (Li^+) and accelerated by a potential difference of 900 V. The accelerated ions enter a uniform magnetic field $B = 0.04$ T, which deflects the beam. After moving along a semi-circle, the ions hit a photographic film. Two spots separated by a distance x appear on the film. Calculate x. Neglect friction, relativistic effects and gravitational force. Useful unit conversion: 1 au $= 1.66 \times 10^{-27}$ kg.

Exercise 3.
(Wire loop/Category II/After training: 15 min)
Using Ampère-Laplace law (Biot-Savart law), calculate the magnetic field $\vec{B}$ along the axis of a circular loop of radius R carrying a current i.

Exercise 4.
(Ribbon/Category I (by following the hint)/After training: 10 min)
The sketch (see Fig. 1) shows the cross section of a very thin ribbon of width w, carrying a uniform current i going into the plane of the sheet. The length of the ribbon is assumed to be infinite. Hint: Use the superposition principle for the solution if you consider the ribbon to be composed of a linear array of infinitely long wires. Their field was calculated in the lecture.

1. Find the norm and direction of the magnetic field $\vec{B}$ at a point P in the ribbon's plane at a distance d. Please make a drawing.
2. Draw the norm of the magnetic field as a function of d, with d going from 0 to ∞, and comment on the behaviour of $\vec{B}$ in both limits.

Exercise 5.
(Current in sheet/Griffiths/Category I)
Find the magnetic field of a uniform surface current $\vec{K} = K\hat{x}$ flowing over the whole $x-y$-plane (see Fig. 2), i.e. an infinitely long and wide sheet of current $\vec{K}$. Hint: Analyze first the expected symmetry and relevant components of the magnetic field.

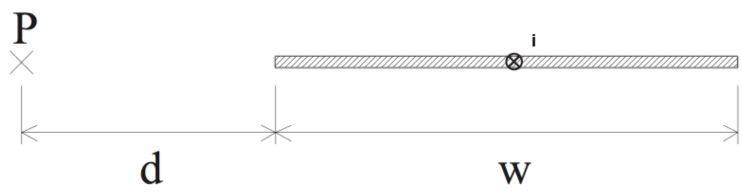


Figure 1: Schematic of the cross-section of a very thin ribbon of width w, carrying a uniform current i going into the plane of the sheet.

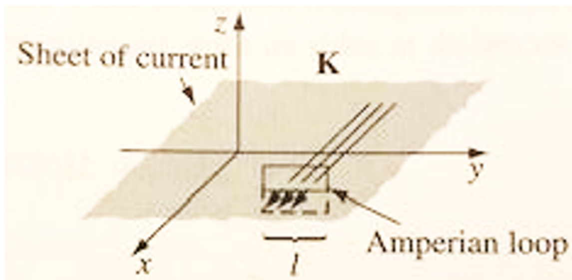


Figure 2: Sheet of current and possible Amperian loop to analyze the magnetic field B after symmetry analysis.

Solution 1.

The force acting on the electrons is the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. To have zero deflection of the electron trajectory means that the acceleration is zero. This, by Newton's law, leads to $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}_0) = \vec{0}$. From here one finds that $\vec{E} = -\vec{v} \times \vec{B}_0$. The settings of the experiment are such that $\vec{v} \perp \vec{B}_0$ hence $|\vec{v} \times \vec{B}_0| = vB_0$. One finds for the magnitude of the electric field that $E = vB_0$. This means that at a fixed setting of the electric field and the magnetic field, the particles that do not deflect from their original trajectory travel with velocity $v = E/B_0$.

Solution 2.

Let us use M, Q and v_M for the particle with mass 7 au and m, q and v_m for the particle with mass 6 au. The process to find the curvature radius is the same for the two objects, it will be shown only once.

The magnetic force $\vec{F}$ acting on the charged particle q is $\vec{F} = q\vec{v} \times \vec{B}$. The magnetic interaction induces a centripetal motion of the object. The centripetal acceleration relating to $\vec{F}$ is $a_c = \frac{v_m^2}{R_m}$, with R_m being the curvature radius. a_c is perpendicular to the tangential velocity v_m and points inward towards the center of the circular trajectory that the particle is describing. It follows that $qv_mB = m\frac{v_m^2}{R_m} \Rightarrow v_m = \frac{R_mqB}{m}$. When the particle enters the zone with uniform magnetic field it has been accelerated with an electrostatic potential thus acquiring kinetic energy: $qV = \frac{1}{2}mv_m^2$.

Combining our knowledge about the system we obtain that $qV = \frac{1}{2}m\left(\frac{R_mqB}{m}\right)^2$. Solving for R_m we conclude that $R_m = \frac{1}{B}\sqrt{\frac{2mV}{q}}$. The same procedure is applied also for the other particle. The separation between the two spots on the photographic film is $x = 2(R_M - R_m)$. Introducing the numerical values we find $x = 2\cdot(0.2858 - 0.2646)\,\mathrm{m} = 0.0424\,\mathrm{m} = 42.4\,\mathrm{mm}$

Solution 3.

From the Ampère-Laplace's (Biot-Savart's) law we derive: $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\vec{u}_t \times \vec{u}_r dl}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \vec{r}}{|\vec{r}|^3}$ (Fig. 3), where ds is the infinitesimal portion of circular loop. Because of symmetry all components off-z-axis cancel out, i.e., $d\vec{B} = dB\hat{z}$. We solve this problem using cylindrical coordinates to exploit the symmetry. Refer to Fig. 3.

$\vec{r}$ is always perpendicular to $d\vec{s}$ therefore $|d\vec{s} \times \vec{r}| = |(ds)r|$. To compute the total magnetic field at a generic point z_0 on the z-axis we calculate $\vec{B} = \int dB\hat{z} = \int \frac{\mu_0 I}{4\pi} \frac{ds}{|\vec{r}|^2} \hat{z} \cos\vartheta$. The term $\cos\vartheta$ is motivated by the fact that we project the vector $d\vec{s} \times \vec{r}$ on the z-axis. It holds $\cos\vartheta = \frac{R}{\sqrt{R^2+z_0^2}}$.

The integral is evaluated over the entire circumference (indicated with Γ) that is described by the current loop: $\oint_\Gamma \frac{\mu_0 I}{4\pi} \frac{ds}{|\vec{r}|^2} \hat{z} \cos\vartheta = \hat{z} \left(_\Gamma \oint \frac{\mu_0 I}{4\pi} \frac{ds}{|\vec{r}|^2} \frac{R}{\sqrt{R^2+z_0^2}} \right)$.

We apply cylindrical coordinates and write $ds = Rd\varphi$, where $d\varphi$ is the angle in the plane of the loop running from 0 to 2π through the entire circular loop.

$\vec{B}(z_0) = \hat{z} \int_0^{2\pi} \frac{\mu_0 I}{4\pi} \frac{Rd\varphi}{R^2+z_0^2} \frac{R}{\sqrt{R^2+z_0^2}} = \hat{z} \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z_0^2)^{3/2}}$

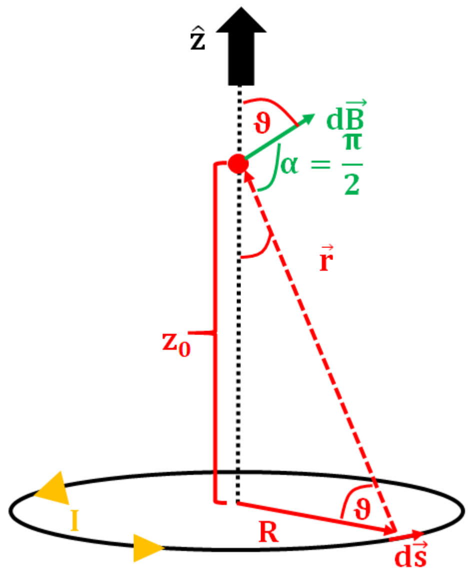


Figure 3: Loop circuit is carrying current I. Geometrical parameters for determining distance and vectors at a generic point along its axis.

Solution 4.

The ribbon can be broken down in strips with infinitesimal width dx. Then we use the *superposition* principle for all these individual infinitely long, very narrow wires. We know the result of the field around such a wire with current I_{wire}: $B_{\text{wire}} = \frac{\mu_0}{2\pi} \frac{I_{\text{wire}}}{x}$ where x is the distance from the wire. Each of the wires carries a current $di = (I/w)dx$. We sum up contributions coming from each of these parallel wires to find the magnetic field in point $\vec{P}$. The drawing of the system is illustrated in Fig. 4.

1. $B = \frac{\mu_0}{2\pi} \int_{Ribbon} \frac{di}{x} = \frac{\mu_0 I}{2\pi w} \int_d^{w+d} \frac{dx}{x} = \frac{\mu_0 I}{2\pi w} ln\left(1 + \frac{w}{d}\right)$.
   The field direction is along $\hat{z}$.
2. For fixed width w, the field intensity goes to zero for $\frac{d}{w} \to \infty$ and goes to infinity for $\frac{d}{w} \to 0$.
   The plotted physical quantity in fig. 5 is the normalized field intensity $B_{norm} = B/\left(\frac{\mu_0 I}{2\pi w}\right) = ln\left(1 + \frac{1}{d/w}\right)$.

Solution 5.

The sketch of the problem is represented in Fig. 6. By Ampère-Laplace's (Biot-Savart's) law it follows that the magnetic field can not have any x-component, i.e. $\vec{B} \cdot \hat{x} = 0$.

By symmetry any contribution along z coming from a filament at $+y$ is cancelled out by the corresponding filament at $-y$. Therefore the z-component of the magnetic field is also zero: $\vec{B} \cdot \hat{z} = 0$.

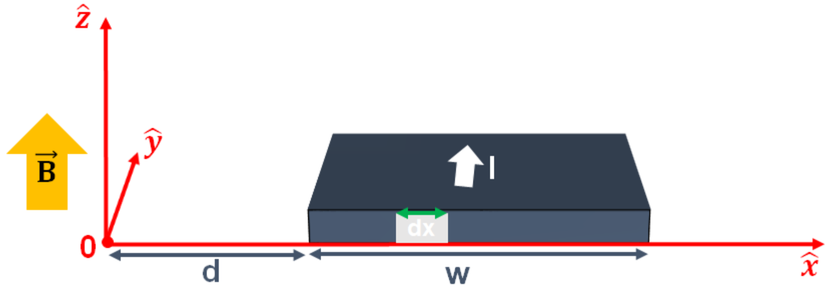


Figure 4: Schematics of the geometry of the problem. The center of the reference framework is the point at which we are interested in measuring the magnetic field. $\vec{B}$ points along z-axis.

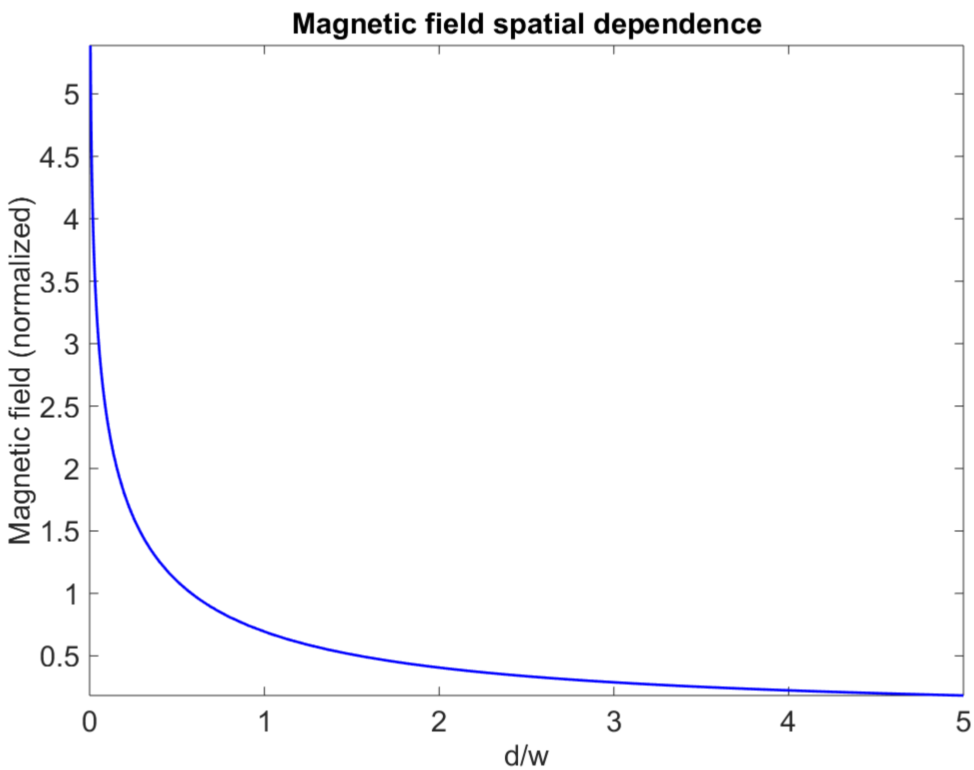


Figure 5: Magnetic field as function of d/w. Magnetic field is normalized before plotting. Normalization factor is $\frac{\mu_0 I}{2\pi w}$

The magnetic field has non-zero y-component $\vec{B} \cdot \hat{y} \neq 0$. We will now find the expression of B_y.

To recap so far we know that the current-carrying sheet generates a magnetic field $\vec{B} = (0, B_y, 0)$.

To calculate B_y we use Ampere's law $\oint_\Gamma \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$ where $d\vec{s}$ follows the chosen Amperian loop along path Γ as sketched in Fig. 7. The amperian loop Γ, that we selected, lies in the y, z-plane and the loop extends both above and below the sheet with a length l as indicated. The integral in the Ampere's law for this case gives $\oint_\Gamma \vec{B} \cdot d\vec{s} = B \cdot (2l)$. In the integral all contributions with $d\vec{s}$ lying along z-axis vanish as we have discussed that $B_z = 0$. Then we reach the result $B_y \cdot (2l) = \mu_0 I_{enc} \rightarrow B_y \cdot (2l) = \mu_0 \cdot (Kl)$.

The magnetic field magnitude is found to be $B_y = \frac{1}{2}\mu_0 K$. This result does not depend on the distance between

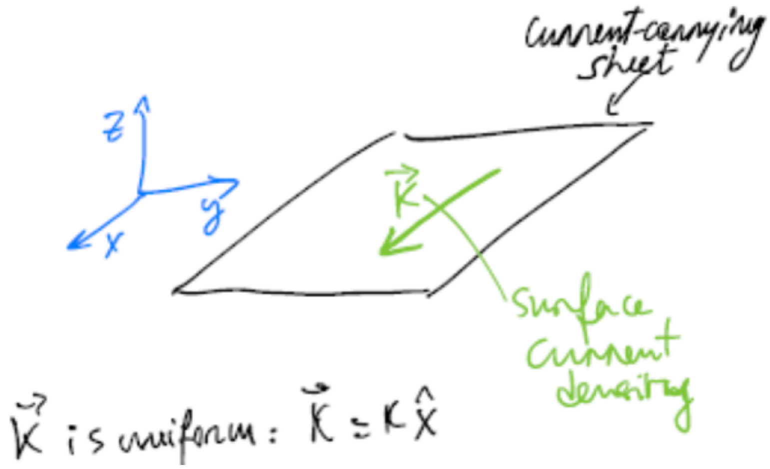


Figure 6: Sketch of problem 5

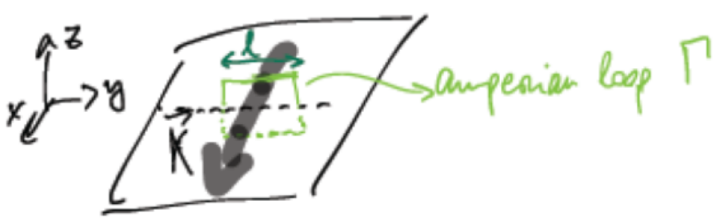


Figure 7: Sketch of the Amperian loop

the current sheet and the position in space (such a uniform vector field was found in electrostatics for the electric field of a uniformly charged infinitely wide plane).

By using the right hand rule one can infer the magnetic direction and conclude that

a) for $z < 0$: $\vec{B} = (+\hat{y})\frac{1}{2}\mu_0 K$

b) for $z > 0$: $\vec{B} = (-\hat{y})\frac{1}{2}\mu_0 K$.