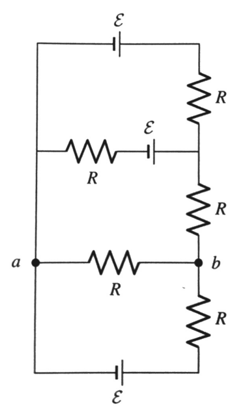
Exercise sheet 7: Currents in circuits with capacitors

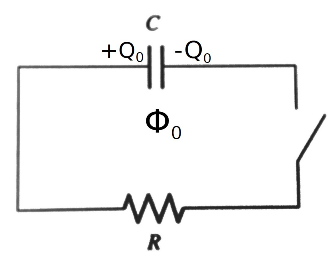
30/10/2024

We indicate the challenges of the problems by categories I (”warming-up”), II (”exam-level”), III (”advanced”). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Kirchhoff's laws - From Purcell/Category I)
What is the potential difference between point a and point b in this circuit? Solve the circuit using Kirchhoff's laws assuming identical batteries for each EMF $\mathcal{E}$ and identical resistors R.



Exercise 2.
(Discharge of a capacitor/category II
(after training about 25 min for solution))



Consider the circuit shown in the sketch. The capacitor with capacitance C is initially charged at a potential difference $\phi_0 = \phi(t = 0)$. At $t = 0$, the switch is closed. The capacitor discharges as a function of time.

a) In which direction does the current flow at $t > 0$? Draw a sketch.

b) Express the potential difference $\phi(t)$ as a function of the charge $Q_c(t)$ on the capacitor and its capacitance C.

c) Express the potential difference $\phi(t)$ as a function of the current $I(t)$ and the resistance R.

d) Use your own words to explain why $I(t) = -\frac{dQ_c(t)}{dt}$.

e) From the equations you have found so far, write the differential equation dictating the evolution of $Q_c(t)$ as a function of time, with R and C as parameters. Solve this differential equation. Don't forget to consider the initial conditions.

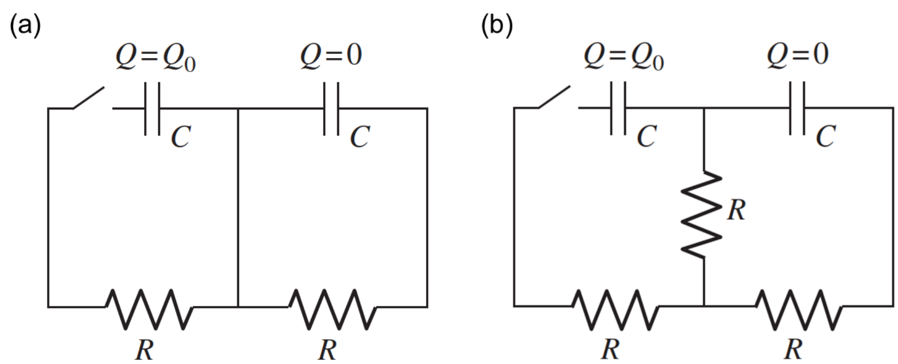
f) How much charge is left in the capacitor at $t = \tau = RC$? τ is the characteristic time of the RC circuit.

g) What was the energy initially stored in the capacitor? What is the energy left after a very long time, say $t >> RC$? Where did it go?

Exercise 3.
(A discharge with two capacitors - From Purcell/Category II)

a) The circuit in (a) contains two identical capacitors and two identical resistors. Initially, the left capacitor has charge Q_0 (with the left plate positive), and the right capacitor is uncharged. If the switch is closed at $t = 0$, find the charges on the capacitors as functions of time. Hint: The loop equations should be simple ones.

b) Answer the same question for the circuit in (b), in which one more identical resistor has been added. What is the maximum (or minimum) charge that the right capacitor achieves? Sketch the time dependencies of the charges Q_1 and Q_2 on the capacitors C_1 (left) and C_2 (right), respectively, and the currents I_1 and I_2 flowing in the two loops. Hint: Find the solutions of the differential equations in that take the sum and difference of the loop equations and solve for the sum and difference of the charges. Then compute each charge individually.



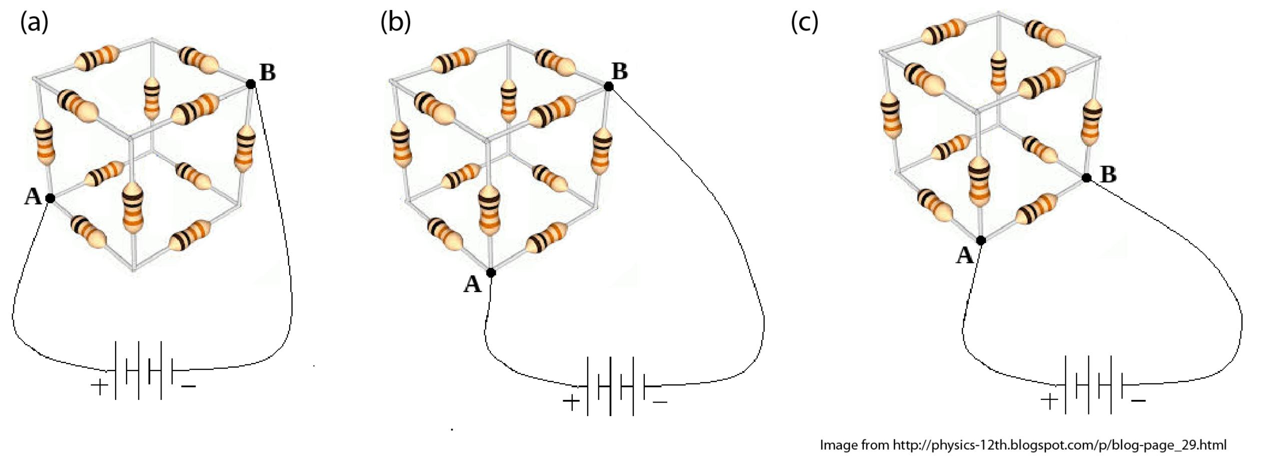
Exercise 4.
(Advanced Question: Cubic network - From Purcell/Category III)
A cube has identical resistors R along each edge. Find the equivalent resistance between two nodes that correspond to:

a) diagonally opposite corners of the cube;

b) diagonally opposite corners of a face;

c) adjacent corners.

You do not need to solve a number of simultaneous equations; instead use symmetry arguments. Hint: Identify nodes that are on the same potential in each different scenario. For the further analysis you can assume that such nodes are interconnected by a perfectly conducting wire.



Solution 1.
The circuit is shown in Fig. 1. At a junction point the total incoming current is equal to the total current exiting the same junction (*Kirchhoff's current law, KCL*). Using this rule we find:

a) $I_1 = I_2 + I_3$

b) $I_3 = I_4 + I_5$

c) $I_6 = I_4 + I_5$

d) $I_1 = I_2 + I_6$

In addition we know that the sum of all potential differences across a complete loop is zero (*Kirchhoff's voltage law, KVL*) (the letter ε represents the emf):

1. $\varepsilon - \varepsilon = I_1R + I_2R$
2. $\varepsilon = -I_2R + I_3R + I_5R$
3. $\varepsilon = -I_4R + I_5R$
4. $\varepsilon = I_1R + I_3R + I_5R$
5. $\varepsilon - \varepsilon = I_1R + I_3R + I_4R$
6. $\varepsilon - \varepsilon = -I_2R + I_3R + I_4R$

Combining different equations one obtains:

- (a) and (d) $\Rightarrow I_3 = I_6$
- (1) $\Rightarrow I_1 = -I_2$
- (a) $\Rightarrow I_1 = I_3/2 = I_6/2$
- (5) $\Rightarrow I_4 = -3I_1$
- (b) $\Rightarrow I_5 = 5I_1$
- (2) $\Rightarrow \varepsilon = 8RI_1$

Therefore we conclude that $V_b - V_a = RI_5 = 5RI_1 = \frac{5}{8}\varepsilon$.

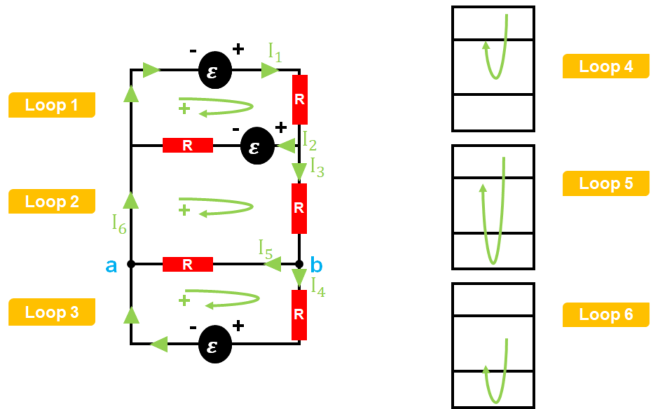


Figure 1: Circuit schematics. Positive direction for current flow is chosen and displayed for each junction.Green arrows represents different currents. All loops inside the circuit are represented.

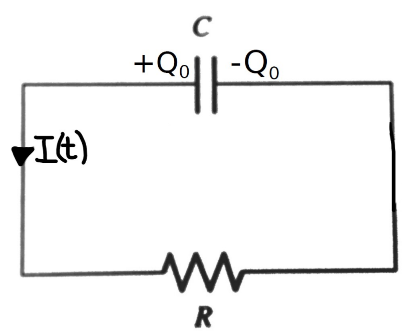


Figure 2: Sketch for direction of current.

Solution 2.

a) Current flows from the positively charged plate to the negatively charged one.

b) $\phi(t) = \frac{Q_c(t)}{C}$

c) $\phi(t) = RI(t) = R\left(-\frac{dQ_c(t)}{dt}\right)$.

d) Charge flows from one capacitor plate to the other and goes through the resistance R. Current I measures the variation of charge flow per unit time through a defined surface. In this case our surface is that of the resistive wire and the charge going in is the charge that left the capacitor. Therefore, $I_R = -\frac{dQ_c(t)}{dt}$.

e) $\phi(t) = RI(t) = -R\frac{dQ_c(t)}{dt} \Rightarrow \frac{Q_c(t)}{C} = -R\frac{dQ_c(t)}{dt}$. We obtain $R\frac{dQ_c(t)}{dt} + \frac{Q_c(t)}{C} = 0$ i.e. the differential equation describing RC circuits. To solve this differential equation we separate the variables to reach this expression $\frac{dQ_c(t)}{Q_C(t)} = -\frac{dt}{RC}$, let us define $\tau = RC$. We integrate both terms $\int_{Q_c(0)}^{Q_c(t')} dQ_c\frac{1}{Q_c} = -\frac{1}{\tau}\int_0^{t'} dt$ to obtain $ln\left(\frac{Q_c(t')}{Q_0}\right) = -\frac{t'}{\tau}$; with $Q_0 = Q_c(0)$. The last equation is inverted to find the temporal evolution in the circuit of the charge: $Q_c(t') = Q_0 \exp\left(-t'/\tau\right)$.

f) $Q_c(\tau) = Q_0 \exp(-1) = \frac{Q_0}{e}$, with e being Euler's number.

g) Initial capacitor energy is $U_c = \frac{1}{2}Q_0\phi_0$, with $\phi(t=0) = \phi_0$. Energy decays exponentially as it is proportional to the stored charge square. Energy is dissipated by the resistor R due to Joule heating: $W_{dissipated} =$

$\int_0^\infty RI(t)^2 dt = \int_0^\infty R\left(-\frac{dQ_c(t)}{dt}\right)^2 dt = \int_0^\infty R\left(-\frac{d(Q_0\exp(-t/\tau))}{dt}\right)^2 dt = \frac{RQ_0^2}{\tau^2}\int_0^\infty \exp(-2t/\tau)dt = \frac{RQ_0^2}{\tau^2}\frac{-\tau}{2}[\exp(-2t/\tau)]_0^\infty = \frac{RQ_0^2}{2\tau} = \frac{Q_0^2}{2C} = \frac{1}{2}Q_0\phi_0$.

Solution 3.

a) Let Q_1 and Q_2 be the charges on the (left plate of the) left and right capacitors. And let I_1 and I_2 be the left and right loop currents, with counterclockwise positive, as shown in Fig. 3. Then the two loop equations are

$$\frac{Q_1}{C} - I_1 R = 0 \qquad \text{and} \qquad \frac{Q_2}{C} - I_2 R = 0.$$

But $I_1 = -dQ_1/dt$ and $I_2 = -dQ_2/dt$, so we have

$$\frac{Q_1}{C} + R\frac{dQ_1}{dt} = 0 \qquad \text{and} \qquad \frac{Q_2}{C} + R\frac{dQ_2}{dt} = 0.$$

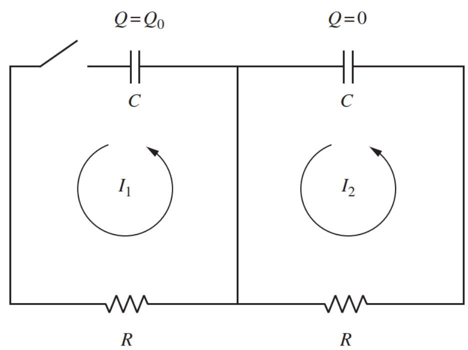


Figure 3: Sketch of the current loops.

These two equations are decoupled, so we can solve for Q_1 and Q_2 separately. Separating variables and integrating each equation, we find that both Q_1 and Q_2 are proportional to $e^{-t/RC}$. Given the initial charges of Q_0 and 0, we see that the charges as functions of time are $Q_1(t) = Q_0 e^{-t/RC}$, and $Q_2(t) = 0$.

b) The two loop equations are now

$$\frac{Q_1}{C} - I_1 R - (I_1 - I_2)R = 0 \qquad \text{and} \qquad \frac{Q_2}{C} - I_2 R - (I_2 - I_1)R = 0.$$

These equations are coupled; they both involve Q_1 and Q_2. If we add them, we obtain:

$$\frac{Q_1 + Q_2}{C} - (I_1 + I_2)R = 0 \Rightarrow \frac{Q_1 + Q_2}{C} + R\frac{d(Q_1 + Q_2)}{dt} = 0.$$

This equation involves only the combination $Q_1 + Q_2$ of the charges. The solution is $Q_1 + Q_2 = Ae^{-t/RC}$, where A is a constant, determined by the initial conditions. Similarly, if we take the difference, we obtain

$$\frac{Q_1 - Q_2}{C} - 3(I_1 - I_2)R = 0 \Rightarrow \frac{Q_1 - Q_2}{C} + 3R\frac{d(Q_1 - Q_2)}{dt} = 0.$$

The solution here is $Q_1 - Q_2 = Be^{-t/3RC}$, where B is another constant. Having solved for $Q_1 + Q_2$ and $Q_1 - Q_2$, we can take the sum and difference of these results to obtain

$$Q_1(t) = ae^{-t/RC} + be^{-t/3RC} \qquad \text{and} \qquad Q_2(t) = ae^{-t/RC} - be^{-t/3RC},$$

where $a \equiv A/2$ and $b/2$. The initial condition $Q_2(0) = 0$ gives $a = b$. The initial condition $Q_1(0) = Q_0$ gives $a = b = Q_0/2$. So the desired charges as functions of time are

$$Q_1(t) = \frac{Q_0}{2}\left(e^{-t/RC} + e^{-t/3RC}\right),$$

$$Q_2(t) = \frac{Q_0}{2}\left(e^{-t/RC} - e^{-t/3RC}\right).$$

From these expression, we find that Q_2 reaches a maximum at some finite time. To find this, we take the derivative of $Q_2(t)$ and set it to zero and obtain $t_{\max} = RC(3/2)\ln 3 \approx 1.65RC$. Plugging this into the expression for $Q_2(t)$ yields a maximum (negative) value of $-Q_0/3\sqrt{3} \approx -0.19Q_0$. The currents are obtained from the negative derivative of the respective charges. The time dependence of the charges and currents is sketched in Fig. 4:

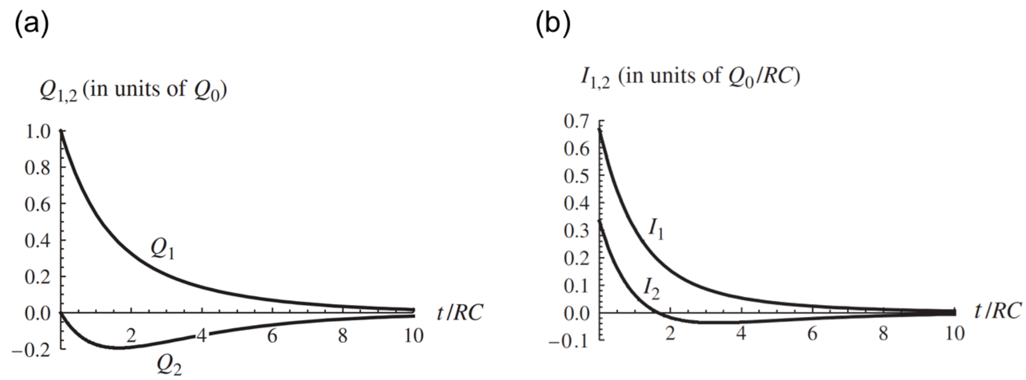


Figure 4: Sketches of the charges (a) and currents (b) as a function of time.

Solution 4.

a) Current is split at beginning in three branches which ends at the same potential since resistors have all equal value. This is equivalent to a one branch with resistance $R_{AC} = R/3$. From point C current can flows towards six different branches causing same voltage drop because the resistance of each branch is identical. Therefore from C we can imagine to have only one connection with resistance $R_{CD} = R/6$. From D according to the cubic arrangement the current has three ways (also with same resistance R) to flow towards point B therefore the equivalent connection has resistance $R_{DB} = R/3$. This is how the electrical circuit is built in Fig. 6. The final equivalent circuit has resistance $R_{tot} = R/3 + R/6 + R/3 = (5/6)R$.

b) Following similar reasoning as in the previous case we identify junction at same potential and we compute equivalent resistance for each of these junctions. We build then the electrical circuit as in Fig. 6b. In this case we notice (Fig. 6b) that $R_{AED} = R_{DFB} = (3/2)R$. Branch $AEDFB$ is replaced by an equivalent branch ADB with a resistance $R_{AD} = (3/2)R$ between A and D and a resistance $R_{DB} = (3/2)R$ between A and D. In this simplified equivalent circuit $R_{AD} = R_{DB}$ and $R_{AC} = R_{CB}$ therefore the potential at points C and D is the same and no current flows in the resistor between D and C. Hence the current flowing from A to E goes directly to F and then B. Then the final total resistance is $\frac{1}{R_{tot}} = \left[\frac{1}{R/2+R/2} + \frac{1}{(3/2)R+(3/2)R}\right] \Rightarrow R_{tot} = (3/4)R$.

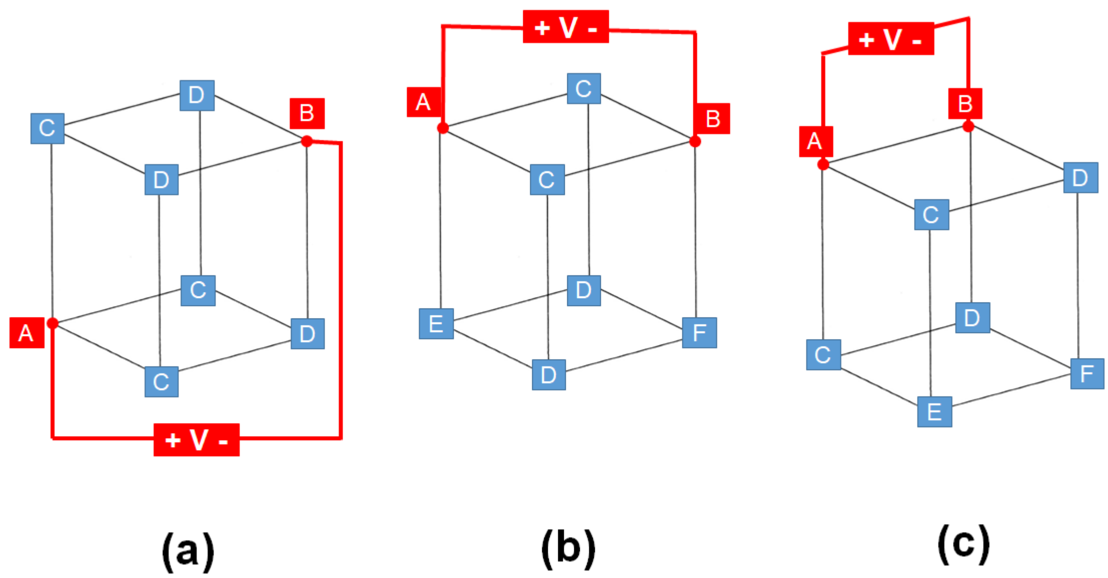


Figure 5: 3D resistor cubic structures with voltage source. Cube vertices at same electric potential are labeled with same letter. At the bottom of the drawings letters (a), (b), (c) make reference to the different cases that are listed in the text of the exercise.

c) Looking at fig. 6c we argue that $R_{CD} = R/2$ and the branch $CEFD$ becomes an equivalent branch with resistance $r = R/2 + R + R/2 = 2R$. This branch is in electric parallel with CD therefore the equivalent resistance is $R_{CD,tot} = \frac{1}{(1/r)+(2/R)} = (2/5)R$. Now the connection from A to B is split in two branches one is AB and one is $ABCD$ and we just computed the total equivalent resistance of branch CD. Therefore $R_{ABCD,tot} = R/2 + R_{CD,tot} + R/2 = (7/5)R$. We conclude that $R_{tot} = R + R_{ABCD,tot} = (7/12)R$.

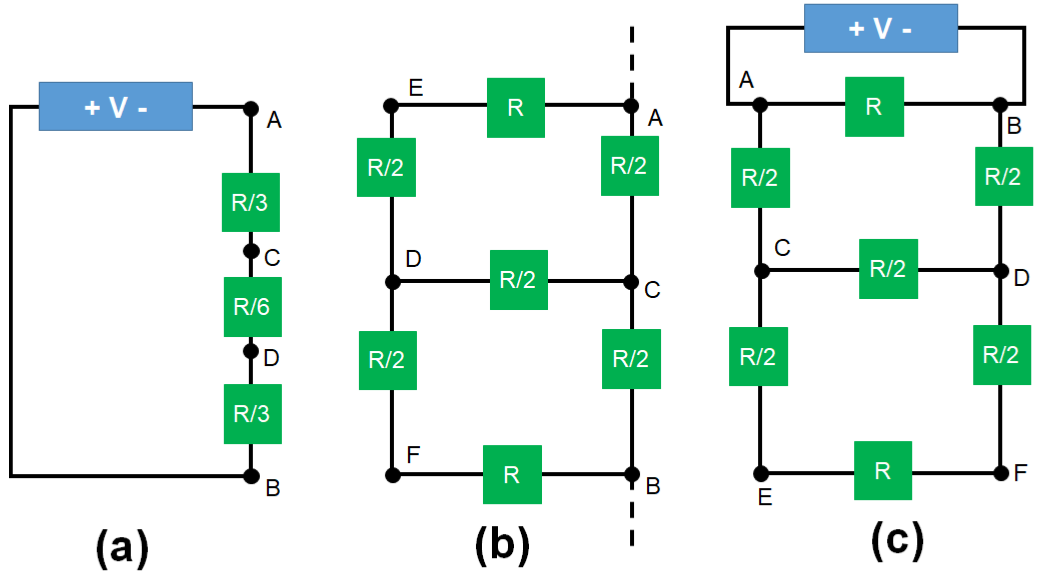


Figure 6: Electric circuit corresponding to different resistor arrangement. At the bottom of the drawings letters (a), (b), (c) make reference to the different cases that are listed in the text of the exercise.