Exercise sheet 6: Screened electric field, energy in capacitor, currents and Ohm's law

16/10/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Extraction of a dielectric slab/Category I)
We consider a parallel plate capacitor with a polystyrene slab ($\epsilon_r = 2.5$) between the plates. Its capacitance is $C = 10$ nF. The dielectric slab is extracted from the capacitor. During the process, the capacitor stays connected to a voltage generator applying a potential difference $\Delta\phi = 100$ V between the plates. Calculate:

a) the variation of the charge on one of the plates,

b) the variation of energy stored in the capacitor,

c) the amount of work needed to take the dielectric slab out.

Exercise 2.
(Current through shell-like metal/Category I)
Two long coaxial cylinders consisting of thin perfect conductors with radii a and b are separated by a metal of conductivity σ (Fig. 1). The two perfect conductors are maintained at a potential difference $\Delta\phi$. What current I flows from one conductor to the other one in a length L? Derive the formula for I as a function of L, σ, a, b, and $\Delta\phi$.

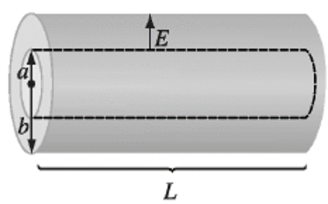
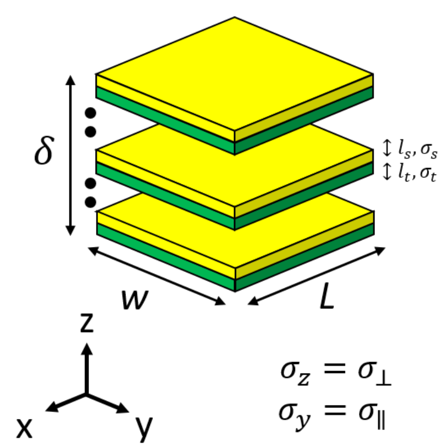


Figure 1: Two long coaxial cylinders consisting of thin perfect conductors with radii a and b are separated by a metal of conductivity σ.

Exercise 3.
(Laminated conductor - From Purcell/Category II/ expected time 35 mins to solve after training)



A laminated conductor is made by depositing, alternately, layers of silver with thickness l_s and conductivity σ_s, and layers of tin with thickness l_t and conductivity σ_t. Assume there are equal layers of silver and tin. The composite material shows different electrical conductivities depending on whether the current is applied in the plane of the layers ($\sigma_{\parallel}$) or perpendicular to them ($\sigma_{\perp}$). The individual conductivities are scalar (isotropic).

a) Find the expression of $1/\sigma_{\perp}$ as a function of σ_s, σ_t, l_s and l_t.
Consider $\sigma_s = 6.3 \times 10^7$ $(\Omega\,\text{m})^{-1}$ and $\sigma_t = 8.7 \times 10^6$ $(\Omega\,\text{m})^{-1}$. Calculate $\sigma_{\perp}$ in the case of $l_s = l_t$. Hint: Assume that the top-most surface is an equipotential

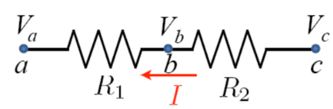
surface and the bottom-most surface is also an equipotential surface but at a different potential value. Evaluate the spatial variation of the electric field in the individual layers.

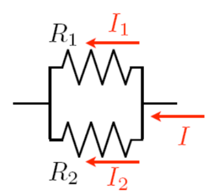
b) Find the expression of $\sigma_{\parallel}$ as a function of σ_s, σ_t, l_s and l_t.
Calculate $\sigma_{\parallel}$ in the case of $l_s = l_t$ considering the parameters of (a). Hint:
In a real experiment the two surfaces parallel to the x, z-plane would be covered by an ultra-thin conductor sheet which are in electrical contact with each layer and represent two equipotential surfaces at the two ends of all layers. Hence each layer experiences the identical potential difference in y-direction.

c) The anisotropy ratio $r_A = \sigma_{\parallel}/\sigma_{\perp}$ can be written as: $\frac{\sigma_{\parallel}}{\sigma_{\perp}} = 1 + \frac{l_s/l_t}{(1+l_s/l_t)^2}\left(\frac{(1+\sigma_s/\sigma_t)^2}{\sigma_s/\sigma_t} - 4\right)$.
At fixed conductivities, how to maximize or minimize the above ratio r_A? You can study the variations of the above function of $x = l_s/l_t$.

d) *Extra Question:* Find the formula for the anisotropy ratio (as given in part (c))

Exercise 4.
(Parallel and series resistors/Category I)





a) Consider the two resistors in series represented in the sketch. By expressing the voltage drop $V_c - V_a$ as a function of the current I flowing through both resistors in series, show that $R_{\text{tot}} = R_1 + R_2$. Calculate R_{tot} for $R_1 = 5\ \Omega$ and $R_2 = 30\ \Omega$. Resistors are illustrated by zig-zag symbols. Remember that the connecting wires between resistors are assumed to be perfect conductors through which charges move according to Newton's first law.

b) Consider the two resistors in parallel represented in the sketch. By expressing the current $I = I_1 + I_2$ as a function of the voltage drop across the resistors, show that $1/R_{\text{tot}} = 1/R_1 + 1/R_2$. Calculate R_{tot} for $R_1 = 5\ \Omega$ and $R_2 = 30\ \Omega$. Remember that the connecting wires between resistors are assumed to be perfect conductors and hence are equipotential lines or surfaces.

c) Consider the parallel resistors again. Assume I splits up in I_1 and I_2 in such a manner that the quantity $P = R_1 I_1^2 + R_2 I_2^2$ is minimized (the quantity P is the dissipated power due to charge scattering/Ohm's law). Express I_1 and I_2 as function of R_1, R_2 and I. Compare with I_1 and I_2 found based on the solution of (b).

d) Calculate the dissipated power P for (a) and (b) in case (i) a current $I = 1$ A or (ii) a potential difference (voltage) $\Delta V = 1$ V is applied.

Solution 1.

a) Potential difference is constant $\Rightarrow \Delta\phi = \frac{Q}{C} = \frac{Q'}{C'}$; primed quantities refer to parameters after removing dielectric slab. Without dielectric between the plates, the capacitance decreases: $C' = \frac{C}{\epsilon_r}$. To conserve potential difference, the accumulated charge must decrease of the same factor as for the capacitance. Therefore $\delta Q = Q' - Q = Q(\epsilon_r^{-1} - 1) = C\Delta\phi(\epsilon_r^{-1} - 1) = -600$ nC.

b) $\delta U = U' - U = \frac{1}{2}C'(\Delta\phi)^2 - \frac{1}{2}C(\Delta\phi)^2 = \frac{1}{2}C\phi^2(\epsilon_r^{-1} - 1) = -30\ \mu$J.

c) Conservation of energy gives that the change in energy of the capacitor equals the total work done. The total work consists of the work done by an external force removing the dielectric and the work done by the battery in transferring charge on the capacitor: $\delta U = W_e + W_b$, with W_e and W_b being respectively the external work and work done by the battery. It follows $W_e = \delta U - W_b = \delta U - (\delta Q)\cdot(\Delta\phi) = 30\ \mu$J.

Solution 2.

Applying the potential difference $\Delta\phi$ will induce a charge per unit length of λ on the inner cylinder. Applying the Gauss's law by considering a cylindrical Gaussian surface with radios $a < r < b$ and length L we obtain the electric field between the cylinders as

$$\oiint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^L \lambda \; dz.$$

Here, we have introduced the linear charge density λ since for $a < r < b$, the electric field produced by the charges on the inner cylinder is equivalent to the field produced by a line charge of equal total charge. Performing the integration gives

$$2\pi r L E = \frac{1}{\varepsilon_0} \lambda L \qquad \Rightarrow \qquad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}.$$

We determine λ as following

$$\Delta\phi = \phi(a) - \phi(b) = -\int_b^a \vec{E} \cdot d\vec{r} = -\int_b^a \frac{\lambda}{2\pi\varepsilon_0 r} dr = -\frac{\lambda}{2\pi\varepsilon_0} \left[\ln(r)\right]_b^a$$

$$\Delta\phi = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right) \qquad \Rightarrow \qquad \lambda = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)} \Delta\phi$$

From Ohm's law we have $\vec{J} = \sigma\vec{E}$, therefore the current is

$$I = \int \vec{J} \cdot d\vec{a} = \int_0^L \sigma\vec{E} \cdot (2\pi r \; dz \; \hat{r}) = 2\pi\sigma \int_0^L \frac{\lambda}{2\pi\varepsilon_0 r} \; r \; dz = \frac{\sigma}{\varepsilon_0}\lambda L$$

$$I = \left(\frac{\sigma L}{\varepsilon_0}\right)\left(\frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}\Delta\phi\right) = \frac{2\pi\sigma L}{\ln\left(\frac{b}{a}\right)}\Delta\phi$$

We point out that the expression derived above for the potential difference $\Delta\phi$ as a function of linear charge density can be rewritten as $\Delta\phi = (\lambda L)\left(\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}\right)^{-1} = \frac{Q}{C}$. Here, $Q = \lambda L$ is the amount of charge on a segment of length L on the cylinder and $C = \left(\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}\right)$ is the capacitance of a cylindrical capacitor of length L (see exercise 4d of exercise sheet 4).

Solution 3.

a) The physical structure is displayed in Fig. 2. The total voltage drop is $\Delta\phi = \Delta V = \int \vec{E} \cdot d\vec{l} = \frac{\delta}{l_s + l_t}(E_s l_s + E_t l_t)$. It is common practice to use ΔV instead of $\Delta\phi$ in the analysis of circuits and Ohm's law. We assume the electric field is uniform across the thickness of each layer. The integral is performed by summing up the voltage drop in one silver layer and in one tin layer (the term between parenthesis) and multiplying this sum by the total number of layers.

Considering that $E = J/\sigma$, we can rewrite the voltage drop as $\Delta V = \frac{\delta}{l_s+l_t}(\frac{J_s}{\sigma_s} l_s + \frac{J_t}{\sigma_t} l_t)$. The numerical factor $\frac{\delta}{l_s+l_t}$ counts the number of layers. We express now the voltage drop in term of total current flowing through the sketched circuit (Fig. 2). Current densities in each material are the same $J_s = J_t = J_{total}$ therefore $J_s = J_t = I_{total}/(WL)$, with $WL = A$ being the area relevant for charge flow. We calculate that $\Delta V = \frac{\delta}{A}\frac{1}{l_s+l_t}\left(\frac{l_s}{\sigma_s} + \frac{l_t}{\sigma_t}\right) I$. Considering both $R = \frac{\delta}{\sigma_\perp A}$ motivated by the general formula for a resistance calculated from a specific conductivity σ and Ohm's law $\Delta V = RI$ we conclude that $\frac{1}{\sigma_\perp} = \frac{1}{l_s+l_t}\left(\frac{l_s}{\sigma_s} + \frac{l_t}{\sigma_t}\right)$.

For $l_s = l_t = l$ it follows that $\frac{1}{\sigma_\perp} = \frac{\sigma_s+\sigma_t}{2\sigma_s\sigma_t}$.

Using the given numerical values we obtain $\sigma_\perp = \frac{2 \cdot 63 \cdot 8.7}{63+8.7} \cdot 10^6 \frac{1}{\Omega m} = 15.3 \cdot 10^6 \frac{1}{\Omega m}$.

b) Electric field in each material is the same i.e. $E_s = E_t = \frac{\Delta V}{w} = E$ (Fig. 3). Moreover $J_{tot} = \sigma_{||} E$ and $J_s/\sigma_s = J_t/\sigma_t = E$. The total current density over the entire cross sectional plane is related to the total

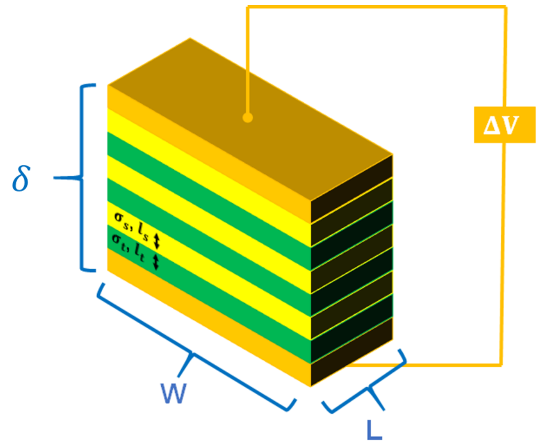


Figure 2: Layered structure of alternately stacked silver and tin films. The sketch shows also the electric configuration for perpendicular electric measurement.

current density in silver and the one in tin: $J_{tot} = \frac{I_{tot}}{\delta L} = \frac{1}{\delta L}\int \vec{J}\cdot d\vec{a} = \frac{1}{\delta L}\left[\left(l_t L \frac{\delta}{l_s+l_t}\right) J_t + \left(l_s L \frac{\delta}{l_s+l_t}\right) J_s\right] = \frac{l_t J_t + l_s J_s}{l_s+l_t} = \frac{l_t\sigma_t + l_s\sigma_s}{l_s+l_t}E$. We conclude that $\sigma_{||} = \frac{l_t\sigma_t+l_s\sigma_s}{l_s+l_t}$.

For $l_s = l_t = l$ it follows that $\sigma_{||} = \frac{\sigma_s+\sigma_t}{2}$. Using the given numerical values one finds $\sigma_{||} = 35.85\cdot 10^6 \frac{1}{\Omega m}$.

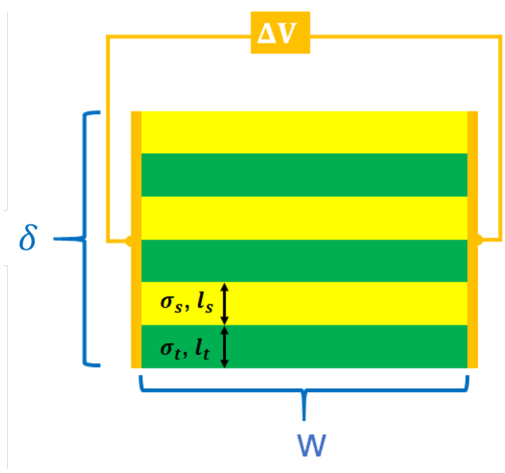


Figure 3: Layered structure of alternately stacked silver and tin films. The sketch shows also the electric configuration for parallel electric measurement. Physical and geometrical parameters are the same as in Fig. 2. L (here not displayed) is the dimension of the system along the interface with the metallic plates for electrical contact.

c) For fixed conductivities the ratio can be maximized or minimized by tuning the ratio between the thickness of the two layers, namely $x = l_s/l_t$. Therefore the function that we need to maximize/minimize is $f(x) = \frac{x}{(1+x)^2}$ and $f(x) > 0$ for any x value. Assuming that $\sigma_s > 0$ and $\sigma_t > 0$ ensures that $r_A > 0$. We note that:

$$f(x) \to 0 \text{ for both cases: } x \to 0, x \to \infty$$

To find the point of maximum x_0 of $f(x)$ we calculate the first derivative $f'(x)$ and set it equal to zero. We solve that equation finding $x = x_0$ and we check that $f''(x = x_0) < 0$.

$$f'(x) = 0 \Leftrightarrow \frac{1-x}{(1+x)^3} = 0 \Rightarrow x = x_0 = 1$$

First derivative becomes zero for $x = x_0 = 1$ we now check that in this point the second derivative is negative

$$f''(x)|_{x=x_0} = \left[\frac{2x-4}{(1+x)^4}\right]_{x=x_0} = \frac{-2}{2^4} < 0$$

Therefore we found a maximum of $f(x)$ for $x = 1$.

The anisotropy ratio is maximized when the two material have same thickness $l_s = l_t = l$. In case the block is made of only one material (i.e. $\sigma_s = \sigma_t$) than the anisotropy ratio is minimized: $\left\{\frac{[(\sigma_s/\sigma_t)+1]^2}{\sigma_s/\sigma_t} - 4\right\} = 0 \Rightarrow$ $r_A = 1$. Therefore in general it holds $r_A \geq 1$.

d) The expression is obtained combining the formula that have been obtained in (a) and (b). The anisotropy ratio is:

$$r_A = \frac{\sigma_{||}}{\sigma_\perp} = \frac{l_t\sigma_t + l_s\sigma_s}{l_s + l_t}\frac{1}{l_s + l_t}\left(\frac{l_s}{\sigma_s} + \frac{l_t}{\sigma_t}\right) =$$

$$\frac{1}{(l_s+l_t)^2}\frac{l_s^2\sigma_s\sigma_t + l_tl_s\sigma_t^2 + l_tl_s\sigma_s^2 + l_t^2\sigma_t\sigma_s}{\sigma_s\sigma_t} = \frac{1}{(l_s+l_t)^2}\frac{l_s^2\sigma_s\sigma_t + l_tl_s\sigma_t^2 + l_tl_s\sigma_s^2 + l_t^2\sigma_t\sigma_s + 2l_sl_t\sigma_s\sigma_t - 2l_sl_t\sigma_s\sigma_t}{\sigma_s\sigma_t} =$$

$$1 + \frac{l_tl_s\sigma_t^2 + l_tl_s\sigma_s^2 - 2l_sl_t\sigma_s\sigma_t}{\sigma_s\sigma_t(l_s+l_t)^2} = 1 + \frac{l_s/l_t}{[1+(l_s/l_t)]^2}\frac{\sigma_s^2 + \sigma_t^2 - 2\sigma_s\sigma_t}{\sigma_s\sigma_t} =$$

$$1 + \frac{l_s/l_t}{[1+(l_s/l_t)]^2}\left(\frac{\sigma_s^2+\sigma_t^2}{\sigma_s\sigma_t} - 2\right) = 1 + \frac{l_s/l_t}{[1+(l_s/l_t)]^2}\left(\frac{\left(\frac{\sigma_s}{\sigma_t}+1\right)^2\sigma_t^2 - 2\sigma_t\sigma_s}{\sigma_s\sigma_t} - 2\right) =$$

$$1 + \frac{l_s/l_t}{[1+(l_s/l_t)]^2}\left(\frac{\left(\frac{\sigma_s}{\sigma_t}+1\right)^2\sigma_t^2}{\sigma_s\sigma_t} - 4\right) = 1 + \frac{l_s/l_t}{[1+(l_s/l_t)]^2}\left\{\frac{[(\sigma_s/\sigma_t)+1]^2}{\sigma_s/\sigma_t} - 4\right\}$$

Solution 4.

a) $V_b - V_a = R_1I$ and $V_c - V_b = R_2I$. Therefore considering the total voltage drop $V_c - V_a = (V_b - V_a) + (V_c - V_b) = R_1I + R_2I \Leftrightarrow V_a - V_c = (R_1 + R_2)I \Rightarrow R_{tot} = R_1 + R_2$.

For the specified numerical case $R_{tot} = 35\ \Omega$.

b) In this system voltage drop ΔV is equal for both resistors. we can build the following system of equations

$$R_1I_1 = \Delta V$$
$$R_2I_2 = \Delta V$$
$$I = I_1 + I_2$$

Solving this system for I one finds $I = V(R_1^{-1} + R_2^{-1})$. Using Ohm's law one concludes that $R_{tot} = \frac{1}{(R_1^{-1}+R_2^{-1})}$.

For the specified numerical case $R_{tot} = 4.3\ \Omega$.

c) We rewrite the total power $P = R_1(I - I_2)^2 + R_2I_2^2$. P has to be derived to find I_2 corresponding to current splitting for minimum dissipation as stated in the text. $\frac{\partial P}{\partial I_2} = -2R_1(I - I_2) + 2R_2I_2$. We find I_2 such that $\frac{\partial P}{\partial I_2} = 0$. The result is $I_2 = \frac{R_1}{R_1+R_2}I$. It follows that $I_1 = \frac{R_2}{R_1+R_2}I$.

Using the solution of (b) we calculate that $R_{tot} = \frac{R_1R_2}{R_1+R_2} = \frac{\Delta V}{I} \Rightarrow I_1 = \frac{\Delta V}{R_1} = \frac{R_{tot}I}{R_1} = \frac{R_2}{R_1+R_2}I$. We conclude that Kirchhoff's law obey to principle of minimum power. Current flows in such a manner to minimize power dissipation.

d) Useful formulas $P = RI^2 = \frac{(\Delta V)^2}{R} = \Delta V I$

a.(i) $P = 35\,\Omega(1\,\mathrm{A})^2 = 35\,\mathrm{W}$

a.(ii) $P = \frac{(1\,\mathrm{V})^2}{35\,\Omega} = 0.03\,\mathrm{W}$

b.(i) $P = 4.3\,\Omega(1\,\mathrm{A})^2 = 4.3\,\mathrm{W}$

b.(ii) $P = \frac{(1\,\mathrm{V})^2}{4.3\,\Omega} = 0.23\,\mathrm{W}$