Exercise sheet 5: Dipole moments, capacitors, dielectrics

9/10/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

For exercises on capacitors the video (link) is instructive.

Exercise 1.

(Potential of a spherical capacitor/Category II)

An initially neutral metallic ball (conductor) of radius $R_1 = 0.1$ m is charged up with a charge Q by connecting the ball to a potential of $\phi_0 = 600$ V with an ultrathin conductor wire (with respect to the electric ground (earth) which is the potential valid at infinity). After the charging, the connection is cut abruptly such that the charges stay on the disconnected ball. Two thin hemispheres (conductors), initially uncharged, of radii $R_2 = 0.11$ m, are taken from infinity and brought in a position such that they form a closed outer spherical shell with the same center as the ball, without touching it. Calculate the potential of the inner ball for the following cases:

a) the hemispheres have no connection to the ground (earth) or the ball,

b) the hemispheres are connected to the ground (earth) using a perfect conductor,

c) the hemispheres are isolated from the ground and from the other ball when they are moved. Then, once in position, they are connected to the ball using a perfect conductor.

Hints:

i) The ground (earth) is defined as a conductor of infinite capacity, such that it can provide/take any amount of charge without changing its potential which is set to zero. $\phi_{\text{ground}} = \phi_\infty = 0$. Something connected to the ground will always have zero potential.

ii) When a system is isolated, it means that no charge can leave it. Thus, its charge does not change. Its potential, however, can change.

Exercise 2.

(Two different dielectrics; category I)

We consider the plate capacitors represented in Fig. 1 which have large surface area S and are separated by a small distance d, brought to the potential difference $\Delta\phi$.

a. Express electric field $\mathbf{E}$, electric displacement $\mathbf{D}$, polarization $\mathbf{P}$, and surface charge density σ_{free} in those capacitors as a function of the distance d, the surface area S, the potential difference $\Delta\phi$, and the dielectric constants ϵ_1 and ϵ_2.

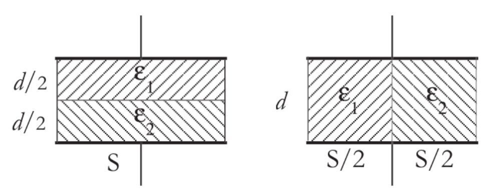

Figure 1: Type 1 (left) and type 2 (right) capacitors.

b. Calculate the absolute values of capacitances of the capacitors for $S = 400$ mm^2, $d = 10$ mm, $\epsilon_1 = 3.4$ (pyrex glass), $\epsilon_2 = 5$ (plexiglass). Refer to Fig. 1.

c. Using the expression obtained in a., sketch the electric field inside the capacitor for the type 1 capacitor. Consider the following cases: (i) $\epsilon_1 < \epsilon_2$, (ii) $\epsilon_1 = \epsilon_2$, and (iii) $\epsilon_1 > \epsilon_2$. What happens at the dielectric interface? Describe the physical origin of what you see.

Exercise 3.
(Insertion of a dielectric slab: isolated capacitor; category I)
A parallel plate capacitor is charged using a voltage generator. The generator is then disconnected, leaving the capacitor charged and isolated. A dielectric slab is then inserted between the conductive plates. Describe qualitatively how the charge Q_{free}, the potential difference $\Delta\phi$, the electric field E, and the capacitance C vary. The capacitance is defined as $C = \frac{Q_{\text{free}}}{\Delta\phi}$. Does the energy stored in the capacitor vary?

Exercise 4.
(Sphere with dielectric shell/Category II)
An insulating solid sphere with radius R is unevenly charged. The positive charge density is described by $\rho = Kr$ where K is a positive constant (with units of A.s/m^4) and $r \leq R$ is the distance (in units of m) from center of the sphere. This problem was discussed in exercise 4 of problem sheet 3. The electrical field outside the orange sphere (Fig. 2) is given in the solution of problem sheet 3. Now the same sphere is surrounded by a homogeneous dielectric shell (blue shell in Fig. 2) extending from $r = R$ to $r = 2R$. Consider an electric permittivity of $\varepsilon_r = 2$. Sketch $E(r)$ relative to the curve provided in in Ex. 4 on sheet 3. Hint: consult the solutions of exercise 4 on problem sheet 3.

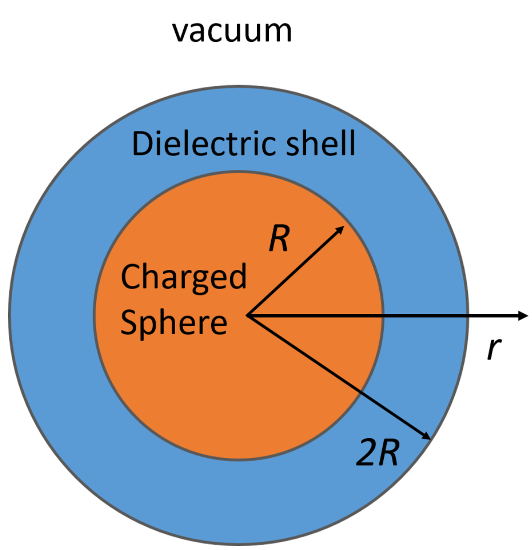

Figure 2: Sketch of the charged sphere with dielectric and vacuum surrounding.

Solution 1.

For a sphere (metallic ball): $E = \frac{Q}{4\pi\epsilon_0 R^2}$. The electric field is in the radial direction.

For the calculation of a capacitance (always positive) of a spherical capacitor one can write $\Delta\phi = -\int_b^a \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$, where Q is assumed to be positive and $b > a$. In this case the result is a positive value.

For the isolated ball, after biasing it to $\phi_0 = 600$ V, the stored charge is $Q = 4\pi\epsilon_0 R_1 \phi_0$. This is found by inverting the formula by which one evaluates ϕ for a charge Q.

a) Hemispheres are isolated from both ground and the ball. The charge Q of the ball induces charge accumulation $-Q$ on the inner wall of the hemispheres.

Consequently $+Q$ accumulates on the outer wall. The total net charge of the shell is zero and there is no modification of voltage ϕ_* of the ball, i.e. $\phi_* = \phi_0$.

b) Hemispheres are grounded. Charge $-Q$ will appear on the inner wall of the hemispheres to counteract the electric field of charge Q of the ball. The modified voltage of the ball is

$\phi_* = -\int_{R_2}^{R_1} \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \phi_0\left(1-\frac{R_1}{R_2}\right) = 54.5$V.

c) Hemispheres are isolated until reaching their final position then they are electrically connected to the metallic ball. To restore equilibrium charge flows from the conductor ball to the (outer surface of) the conductor hemispheres as the outer hemispheres form a Faraday cage and enforce $E = 0$ within this closed shell. The original charge Q is now distributed over a spherical surface with a larger radius. This modifies the capacitance of the interconnected conductors (ball and hemispheres).

The formula for the capacitance of the hemisphere reads $C_h = 4\pi\epsilon_0 R_2$. Once the charge has reached its new equilibrium $C_h = Q/\phi_* \Rightarrow \phi_* = \frac{Q}{4\pi\epsilon_0 R_2} = \phi_0 \frac{R_1}{R_2} = 545.5$ V.

Solution 2.

a. Systems are both illustrated in Fig. 3.

Figure 3: Type 1 and type 2 capacitors.

Type 1. Let us consider Gaussian surfaces enclosing an interface with a metallic plate for both dielectrics separately and the surface of the closed cylindrical volume inside each dielectric and parallel to the interface amounts to A_α. From Gauss's law in dielectric matter we write $\oiint \vec{D}_\alpha \cdot d\vec{a} = \int_{A_\alpha} \vec{D}_\alpha \cdot d\vec{a} = \int_{A_\alpha} \sigma_{free,\alpha} da$, with $\alpha = 1, 2$ being the index referring to dielectric 1 and 2. Integrals are solved to find: $D_1 A = \sigma_{free,1} A$ and $-D_2 A = \sigma_{free,2} A$. The surface charge density of each plate is $\pm\sigma_{free} = \frac{\pm Q_{free}}{S}$ according to the potential difference. The surface densities of free charges for dielectric 1 and 2 are $\sigma_{free,1} = +\sigma_{free}$ and $\sigma_{free,2} = -\sigma_{free}$, respectively. We find that $D_1 = D_2 = D = \sigma_{free}$.

The voltage drop across the capacitor equals the potential difference across the capacitor. We assume $\Delta\phi$ is positive. Then, the potential difference is $\Delta\phi = -\int \vec{E}\cdot d\vec{l} = E_1\frac{d}{2} + E_2\frac{d}{2} = \left(\frac{D_1}{\epsilon_1\epsilon_0} + \frac{D_2}{\epsilon_2\epsilon_0}\right)\frac{d}{2} = \frac{dD}{2\epsilon_0}(\epsilon_1^{-1}+\epsilon_2^{-1}) \Rightarrow D = \frac{2\epsilon_0\Delta\phi/d}{(\epsilon_1^{-1}+\epsilon_2^{-1})} = \sigma_{free}$. Here, $\vec{E}$ and $d\vec{l}$ have been taken anti-parallel which follows from the assumption of positive $\Delta\phi$. Also $\vec{D}_\alpha = \epsilon_\alpha\epsilon_0\vec{E}_\alpha$ ($\alpha = 1,2$) holds. Solving for E_α one finds $E_{1,2} = \frac{2\Delta\phi}{d}\frac{\epsilon_{2,1}}{\epsilon_1+\epsilon_2}$. For the polarization vector in each dielectric $\vec{P}_{1,2}$ the relation with the electric field is exploited: $\vec{P}_{1,2} = (\epsilon_{1,2}-1)\epsilon_0\vec{E_{1,2}} \Rightarrow P_{1,2} = \frac{2\Delta\phi\epsilon_0}{d}\frac{\epsilon_{2,1}(\epsilon_{1,2}-1)}{\epsilon_1+\epsilon_2}$.

The capacitance of such system reads $C_{type1} = Q/\Delta\phi = (\sigma S)/\Delta\phi = (DS)/\Delta\phi = \frac{2\epsilon_0 S/d}{(\epsilon_1^{-1}+\epsilon_2^{-1})}$. This is equivalent to two capacitors with the same plate surface areas and plate spacing $d/2$ that are connected in series. Indeed, $C_{type\,1}^{-1} = \frac{d}{2S\epsilon_0}(\epsilon_1^{-1}+\epsilon_2^{-1}) = C_1^{-1}+C_2^{-1}$.

Type 2. Voltage drop is constant therefore $\Delta\phi = E_1 d = E_2 d \Rightarrow E_1 = E_2 = E = \Delta\phi/d$. Using already mentioned relations one obtains $D_{1,2} = \epsilon_{1,2}\epsilon_0\frac{\Delta\phi}{d}$ and $P_{1,2} = (\epsilon_{1,2}-1)\epsilon_0\frac{\Delta\phi}{d}$. For the total charge Q we calculate that $\sigma_\alpha = D_\alpha$ therefore $Q = S_1\sigma_1 + S_2\sigma_2 = (S/2)(\sigma_1+\sigma_2)$.

The capacitance for this structure is $C_{type\,2} = Q/\Delta\phi = \frac{(S/2)(\sigma_1+\sigma_2)}{\Delta\phi} = \frac{\epsilon_0 S}{2d}(\epsilon_1+\epsilon_2) = \frac{\epsilon_1\epsilon_0 S}{2d} + \frac{\epsilon_2\epsilon_0 S}{2d} = C_1 + C_2$ that is equivalent to two capacitors connected in parallel configuration.

b. Using the numerical parameters of the text calculations yield $C_{type\,1} = 1.43\,\mathrm{pF}$ and $C_{type\,2} = 2.97\,\mathrm{pF}$.

c. Using the expression for the electric field obtained in a., one finds for the electric field inside the capacitor the plot shown in Fig. 4. For this plot, we have assumed arbitrary values for the dielectric constants that cover the three cases. For the case $\epsilon_1 \neq \epsilon_2$, there is a discontinuity at the dielectric interface at $d/2$, with the electric field increasing or decreasing depending on the relative sizes of the dielectric constants. The origin of this jump in electric field is the presence of bound charge at the interface.

The accumulated bound charge Q_{bound} present at the interface is due to e.g. the polarization of the molecules (acting as small dipoles) within each dielectric. Their orientation is induced by the electric field that is generated by surface charges of the capacitor. Depending on the direction of the electric field, the bound charge Q_{bound} at the interface of the two dielectrics can be either positive for $\epsilon_2 < \epsilon_1$ or negative for $\epsilon_2 > \epsilon_1$.

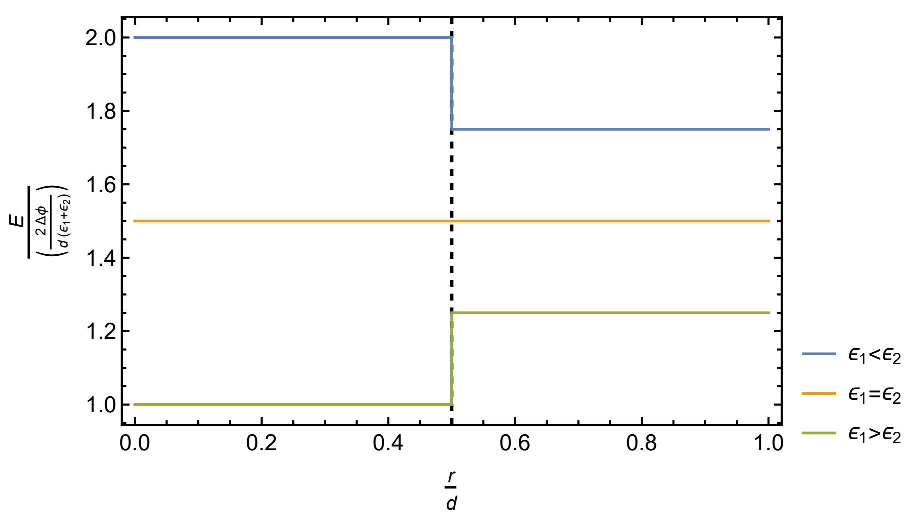

Figure 4: The electric field $\vec{E}$ as a function of position r, both expressed in dimensionless units. $r < 0.5$ corresponds to dielectric ϵ_1 and $r > 0.5$ corresponds to dielectric ϵ_2.

Solution 3.

The capacitor is biased to a fixed voltage by a voltage generator. After disconnecting the system and isolating it, the accumulated charge Q_{free} is kept. Inserting a dielectric material between the two plates does not alter the charge.

The charge produces an electric field that acts as an external field for the dielectric: $\vec{E}_{\text{Q}}$. The insertion of the dielectric modifies the overall electric field in the capacitor, $\vec{E}$, as the dielectric becomes polarized.

Due to material polarization, the electric field inside the capacitor (i.e., inside the dielectric) is $\vec{E} = \vec{E}_{\text{Q}} + \vec{E}^{\text{pol}}$, where $\vec{E}^{\text{pol}}$ is the electric field that is induced by the polarization of the dielectric.

Due to this polarization of the dielectric, the randomly distributed polar molecules rearrange in an ordered manner according to the sensed electric field (see Fig. 5). Now, because of molecular re-arrangement, the induced field in the material counteracts the external field. This field, $\mathbf{E}^{\text{pol}}$, originates from the bound charges induced by the polarization, $\vec{P}$, on the surfaces of the dielectric, and it is anti-parallel to $\vec{E}_{\text{Q}}$.

Therefore, the field $\vec{E}$ inside the capacitor decreases.

We conclude that the voltage drop across the capacitor plates decreases, as this value is proportional to the field: $\Delta\phi = Ed$.

The capacitance $C = \frac{Q_{\text{free}}}{\Delta\phi}$ is inversely proportional to the voltage drop and therefore it increases.

The electrostatic energy density is $U/\text{V} = \frac{\epsilon_0 |\vec{E}|^2}{2}$. After inserting the dielectric layer, the E decreases and hence U/V decreases.

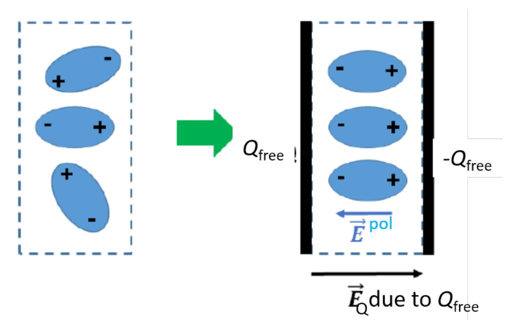

Figure 5: The sketch shows the arrangement of the molecules with a dipole moment in a dielectric when it is inserted between two polarized plates.

Solution 4.

In the presence of a dielectric shell, solutions for the sphere surrounded by vacuum (exercise 4 from exercise sheet 3) hold inside the charged sphere ($r \leq R$) and outside the dielectric shell ($r > 2R$), i.e.,

$$E(r \leq R) = \frac{Kr^2}{4\varepsilon_0}$$

$$E(r > 2R) = \frac{KR^4}{4\varepsilon_0 r^2}$$

For $R < r < 2R$ we need to use the Gauss's law in the presence of a dielectric (for displacement vector $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$), $\oiint \vec{D}.d\vec{a} = 4\pi r^2 D = Q_{enc}$.

$$E = \frac{KR^4}{4\varepsilon_0 \varepsilon_r r^2} = \frac{KR^4}{8\varepsilon_0 r^2}$$

The electric fields for when the sphere is surrounded by vacuum or a dielectric are sketched together in Fig. 6.

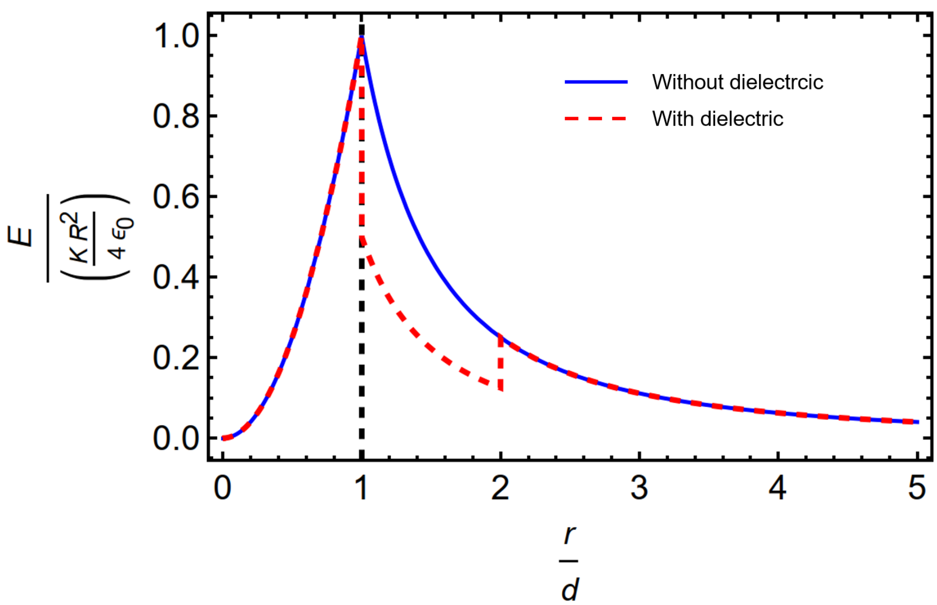

Figure 6: Sketch of the electric field of a non-uniformly charged sphere surrounded by vacuum (Blue) and a dielectric (red).