Exercise sheet 4: Capacitors, Capacitance

2/10/2024

We indicate the challenges of the problems by categories II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Gauss's Law with a Cube) (Category I)

A point charge with charge Q is located at the corner of a cube (Fig. 1). Find the flux through the shaded area (Fig. 1) in terms of Q/ϵ_0.

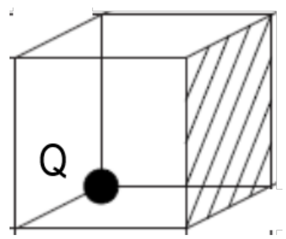

Figure 1: The point-like charge Q is exactly at the corner of a cube.

Exercise 2.

(Gauss's Law with Spheres and Shells) (Category I)

Consider the following four configurations (Fig. 2): (i) a point charge (ii) a sphere of radius r_0 with constant charge density ρ (iii) a sphere of radius r_0 with inhomogeneous charge density $\rho = \rho(r)$ and (iv) a thin spherical shell of radius r_0 with constant charge density ρ. For each case, a total charge Q is considered. Find the electric field at point r_1 in terms of Q and r_1 for each configuration. Assume $r_1 > r_0$.

Exercise 3.

(Application of the Laplace equation/Category II)

We consider two concentric spherical shells made of a conductor with negligible thickness. The inner shell has a radius R_1, the outer one a radius R_2. The inner shell is at potential $\phi_1 > 0$ while the outside one is connected to the ground with a conductor wire, i.e., $\phi_2 = 0$. The potential at infinity is chosen to be zero. To calculate the electric field in such a scenario, one uses the so-called Laplace equation $\vec{\nabla}^2\phi = 0$. This equation is consistent with the Coulomb law and needs to be integrated by considering the boundary conditions.

a) Make a drawing of the problem. Integrate the Laplace's equation between the two spherical shells to find the potential $\phi(\vec{r})$. Draw this potential.

b) Find the electric field from $\phi(\vec{r})$ and draw appropriate vectors.

c) Find the total charge in each spherical shell.

d) The capacitance is defined as $C = \frac{Q}{\phi_1 - \phi_2}$. Calculate C.

Hint: Look into Mathematical Tool Box 1, look at the functional form of $\nabla^2 f$ and integrate over the relevant coordinate.

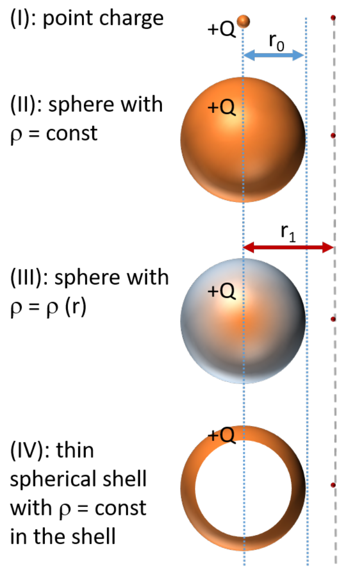

Figure 2: Four configurations, each carrying a total charge Q.

Exercise 4.
(Energy density and capacitance of cylindrical coordinate/Category II; 20 mins are expected for the solution after training for the written exam)
Consider two infinitely long concentric conductor cylinders with radius a and b. They form a capacitor: The charge on the inner cylinder is $+Q$. By choosing the adequate coordinate system:

a) Find the electric field $\vec{E}$ between the two conductors as a function of the relevant charge density σ.

b) Calculate the potential difference between the two cylinders.

c) Calculate the total energy per length, U/L, using $\frac{U}{L} = \frac{1}{L}\int \frac{\varepsilon_0|\vec{E}|^2}{2} dV$.

d) What is the capacitance per length, C/L?

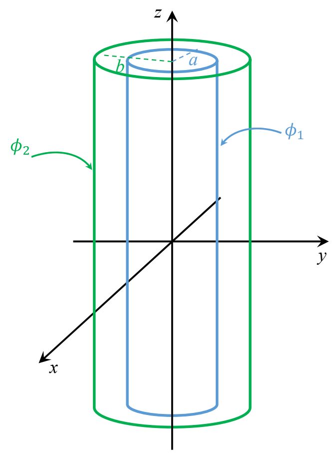

Figure 3: Sketch of the cylindrical equipotential surfaces.

Solution 1.

We refer to Fig. 4.

The approach to solve this problem focuses on finding a Gaussian surface that contains the shaded area and that exhibits an appropriate symmetry with the charge Q residing in the center. This Gaussian surface is then used to evaluate Gauss's law. The large cube is the Gaussian surface that we propose to solve the problem. The cube has 6 facets and each of these corresponds to 4 times the shaded squared area. Therefore in total the cube has a surface that is 24 times the shaded area. All areas exhibit a similar configuration with respect to the charge Q, hence their individual fluxes have the same magnitude of $\Phi = \frac{Q}{24\epsilon_0}$. This value is the solution of the problem.

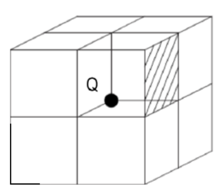

Figure 4: Sketch of the Gaussian surface to solve the problem.

Solution 2.

We note that $r_1 > r_0$ for every proposed configuration. Gauss's law is $\oiint \vec{E} \cdot d\vec{a} = Q/\epsilon_0$. The electric field in each configuration has spherical symmetry hence it depends only on the radial component. Because $r_1 > r_0$ the charge Q that enters Gauss's law is the total charge of each sphere which is the same for all configuration. From this analysis we conclude that the electric field for $\vec{r} = \vec{r}_1$ is the same in each configuration. The result is found using Gauss's law and it is $\vec{E}(\vec{r}_1) = \frac{Q}{4\pi\epsilon_0 r_1^2}\hat{r}_1$

Solution 3.
The Laplacian in spherical coordinates reads:

$$\nabla^2\phi = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

a) A drawing of the problem is given in Fig. 5. The system has spherical symmetry i.e. there is no dependence on angular coordinates φ and θ which are the azimuthal and polar angles respectively. Quantities depend only on the radial distance R from the center.

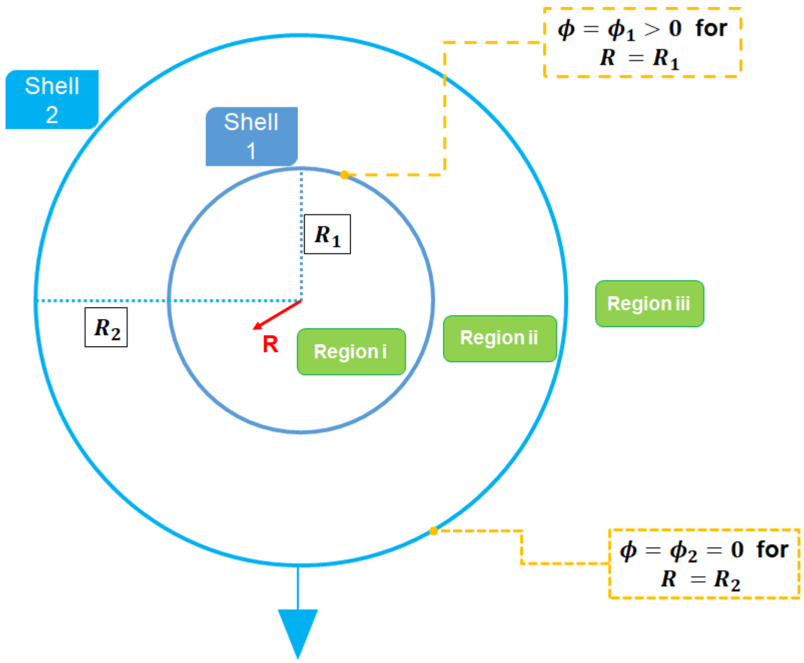

Figure 5: Two concentric spherical shells are analyzed in this exercise. Inner shell is at positive non-zero electric potential ϕ_1. The outer shell is grounded i.e. $\phi_2 = 0$. R_1 and R_2 are the radii of the smallest and the biggest shell. The red arrow represents the radial component R.

We need to identify three regions: (i) $R < R_1$, (ii) $R_1 < R < R_2$, (iii) $R > R_2$. The region asked in the problem was (ii). We discuss here the complete solution:

- Within any volume in region (i) the charge density is always zero. Therefore, Laplace's equation reads $\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) = 0$.
  Electric field is zero for $R < R_1$ (Gauss's law). In addition the problem imposes $\phi(R = R_1) = \phi_1 > 0$. A solution ϕ_i that meets the physical requirements of our system in region (i) is a constant solution $\phi_i = \phi_1$.
- In region (ii) the charge density is zero. Charge is accumulated only on the spherical shell. At the two boundaries of region (ii) $\phi(R_1) = \phi_1 > 0$ and $\phi(R_2) = \phi_2 = 0$. Solving Laplace's equation yields the following general solution:

$$\begin{aligned}\nabla^2\phi &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) = 0\\ \Rightarrow &\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) = 0\end{aligned} \tag{1}$$

By integrating, we have,

$$\left(R^2 \frac{\partial \phi}{\partial R}\right) = A \Rightarrow \left(\frac{\partial \phi}{\partial R}\right) = \frac{A}{R^2} \tag{2}$$

Second integration gives,

$$\phi(r) = -\frac{A}{R} + B \tag{3}$$

Here, A and B are constants of integration. Using the boundary conditions, $\phi(R_1) = -\frac{A}{R_1} + B = \phi_1$ and $\phi(R_2) = -\frac{A}{R_2} + B = 0$, and solving for A and B yields
$A = -\frac{\phi_1 R_1 R_2}{R_2 - R_1}$ and $B = -\frac{\phi_1 R_1}{R_2 - R_1}$.
In region (ii) the electric potential decays as $\propto R^{-1}$.

- In region (iii) charge is absent as well. There is no external electric field and we know from system condition that $\phi(R = R_2) = 0$. Moreover ϕ has to vanish for $R \to \infty$.
  Therefore in region (iii) the solution of Laplace's equation is a constant solution with zero value, i.e. $\phi(R > R_2) = \phi(R = R_2) = 0$.

The electric potential of the system is illustrated in Fig. 6

b) The electric field is $\vec{E} = -\vec{\nabla}\phi$. Electric field is zero in region (i) and (iii), where the electric potential is constant. For region (ii)

$$\vec{E} = -\frac{A}{R^2}\ \hat{R} = \left(\frac{R_1 R_2 \phi_1}{R_2 - R1}\right)\frac{\hat{R}}{R^2}.$$

The spatial dependence of the electric field is shown in Fig. 6.

c) Charge is distributed over two spherical shells. From Gauss's law one finds locally
$E_\alpha = \frac{\sigma_\alpha}{\epsilon_0}$ with $\alpha = 1, 2$ indicating the two shells (Fig. 5).

The electric field direction follows the radial unit vector $\hat{\mathbf{R}}$. The magnitude of the electric field can be evaluated through $E_\alpha = \vec{E}(R = R_\alpha) \cdot \hat{\mathbf{R}}$. The total charge on each shell can then be written as:

$Q_\alpha = \epsilon_0 4\pi R_\alpha^2 [\vec{E}(R = R_\alpha) \cdot \hat{\mathbf{R}}]$, with $\alpha = 1, 2$ indicating the two shells (Fig. 5).

The field points towards the inner side of shell 2, i.e., $\vec{E}$ is antiparallel to the normal vector of the relevant surface. Hence, $\sigma < 0$. This can be shown in Gauss's law. Instead of $E = \sigma/\epsilon_0$ one can write $\vec{E} \cdot \hat{n} = \sigma/\epsilon_0$.

Therefore one finds $Q_1 = -Q_2 = Q = \epsilon_0 4\pi\phi_1 \frac{R_1 R_2}{R_2 - R_1}$.

Q_2 is induced by Q_1.

d) Capacitance $C_s = \frac{Q}{\phi_1 - \phi_2} = \frac{\epsilon_0 4\pi\phi_1 \frac{R_1 R_2}{R_2 - R_1}}{\phi_1} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}$

Solution 4.

From Mathematical Tool Box I, the Gradient operator in cylindrical coordinates is as follows:

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\ \hat{r} + \frac{1}{r}\ \frac{\partial \phi}{\partial \varphi}\ \hat{\varphi} + \frac{\partial \phi}{\partial z}\ \hat{z}.$$

a) Using Gauss's law, we draw a Gaussian cylinder with a radius r, such that $a < r < b$. From the symmetry of the system, the Electric field is only in the radial direction. Given a charge density of σ, from Gauss law:

$$E \cdot (2\pi r l) = \frac{\sigma(2\pi a l)}{\epsilon_0} \Rightarrow \vec{E} = \frac{\sigma a}{\epsilon_0 r}\hat{r} \tag{4}$$

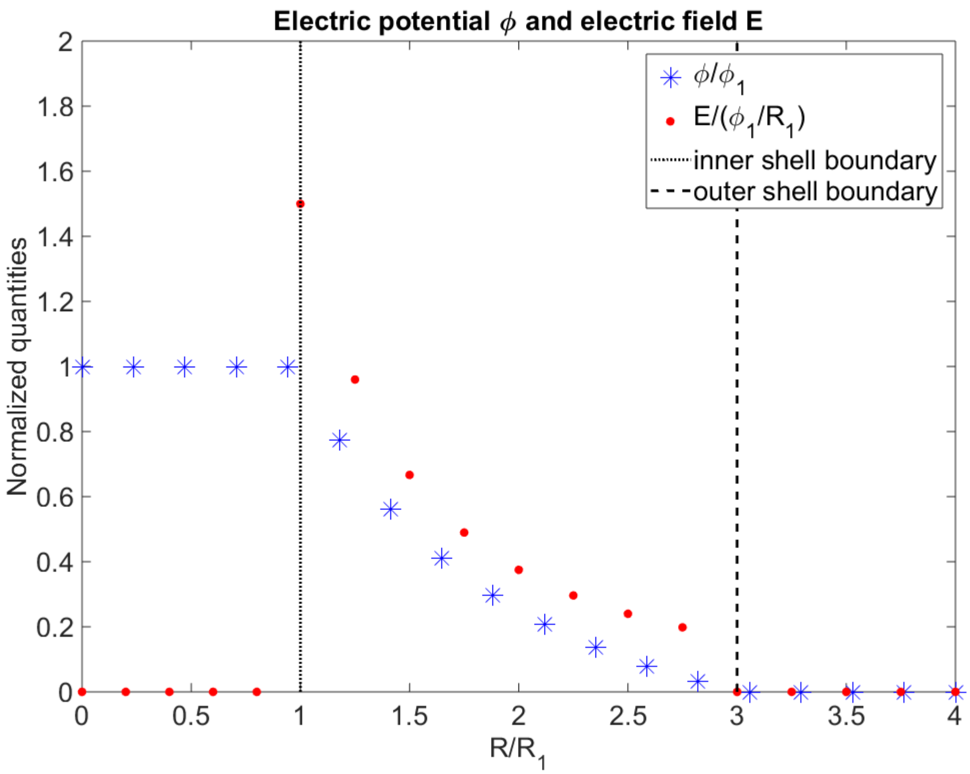

Figure 6: Electric potential ϕ and electric field E are plotted as function of $\frac{R}{R_1}$. Electric potential is normalized to ϕ_1 and the electric field is normalized to $\frac{\phi_1}{R_1}$. For this plot $R_2/R_1 = 3$ and R_1 has been set to 1 (arbitrary scale unit); this plot aims at showing mainly the qualitative behavior of ϕ and E

b) Due to the azimuthal symmetry of the problem, the potential does not depend on φ so: $\frac{\partial \phi}{\partial \varphi} = 0$. Since the cylinders are infinitely long and each of them represents an equipotential surface, ϕ does not depend on z and: $\frac{\partial \phi}{\partial z} = 0$. Therefore, using $\vec{E} = -\vec{\nabla}\phi$, the potential difference can be found as

$$\int_a^b \vec{E} \cdot d\vec{r} = -\int_a^b \vec{\nabla}\phi \cdot d\vec{r} \tag{5}$$

Here, the direction of the line element $d\vec{r}$ can be chosen parallel or antiparallel with $\vec{E}$ but has to be consistent on both sides of the equation. Proceeding with the integration gives:

$$\begin{aligned} \int_a^b \frac{\sigma a}{\epsilon_0 r}\hat{r} \cdot d\vec{r} &= -(\phi(b) - \phi(a)) \\ (\phi(b) - \phi(a)) &= -\frac{\sigma a}{\epsilon_0} \ln \frac{b}{a} \\ (\phi_2 - \phi_1) &= -\frac{\sigma a}{\epsilon_0} \ln \frac{b}{a} \end{aligned} \tag{6}$$

c) The total energy density stored between the cylinders is

$$\frac{1}{2}\varepsilon_0 |\vec{E}|^2 = \frac{1}{2}\varepsilon_0 (\frac{\sigma a}{\epsilon_0 r})^2 \tag{7}$$

By integrating over a length of L of the capacitor, we obtain the total energy per length of the capacitor as

$$\begin{aligned}\frac{U}{L} &= \frac{1}{L}\int_a^b rdr\int_0^{2\pi}d\varphi\int_0^L dz\frac{1}{2}\varepsilon_0(\frac{\sigma a}{\varepsilon_0 r})^2 \\ &= \frac{1}{L}\frac{1}{2}\varepsilon_0(\frac{\sigma a}{\varepsilon_0})^2\int_a^b r\frac{1}{r^2}dr2\pi L \\ &= \pi\varepsilon_0(\frac{\sigma a}{\varepsilon_0})^2\ln\frac{b}{a} \\ &= \pi\varepsilon_0\frac{(\phi_2-\phi_1)^2}{\ln\frac{b}{a}}\end{aligned} \tag{8}$$

d) The energy stored in a capacitor is given by $U = \frac{1}{2}C(\phi_2 - \phi_1)^2$. Therefore, the energy stored per unit length of the cylindrical capacitor is given by $\frac{U}{L} = \frac{1}{2}\left(\frac{C}{L}\right)(\phi_2 - \phi_1)^2$. So, we obtain the capacitance per length, $\left(\frac{C}{L}\right)$, of the cylindrical capacitor as

$$\left(\frac{C}{L}\right) = \frac{2}{(\phi_2-\phi_1)^2}\left(\frac{U}{L}\right) = \frac{2}{(\phi_2-\phi_1)^2}\frac{\pi\varepsilon_0\left(\phi_2-\phi_1\right)^2}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)} \tag{9}$$