Exercise sheet 3: Fields and Potentials, Gauss's Law

25/09/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

For exercise 1 it is instructive to first watch the video (link) and slides containing the solution for an electric field of a charged half-sphere. There we outline how to first analyze symmetries in a 3D scenario and second perform an integration considering ring-like elements.

Exercise 1.
(Potential and Electric Field of a Charged Circle) (Category I)

a) Find the expression for the potential ϕ at a height z over the center of a circle consisting of a uniformly charged line as shown in Fig. 1. The general formula for the potential is $\phi(\vec{r}) = \int \frac{\rho(\vec{r'})dV}{4\pi\epsilon_0|\vec{r}-\vec{r'}|}$. Hint: Consider that the line is a 1D charge distribution.

b) Find the electric field at the same point.

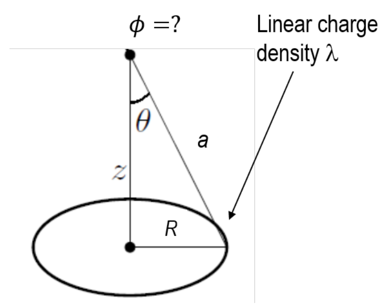

Figure 1: Only the line contains the charges. The black points indicate the center of the loop (bottom) and the position (top) at which the potential ϕ needs to be evaluated.

Exercise 2.
(Charge Density from Electric field) (Category I)

We consider an electric field given by $\vec{E}(\vec{r}) = kr^3\hat{r}$ in spherical coordinates. k is a positive constant and $\hat{r}$ is the unit vector.

a) Which units does the constant k have?

b) Find the expression for the corresponding charge density ρ. How does it depend on the radial distance r?

c) Sketch $\rho(r)$ as a function of radial distance along a line passing through the origin and in a plane intersecting the origin.

d) How large is ρ at $r = 1$ cm if the electric field amounts to $E = 5$ kV/m at the same position?

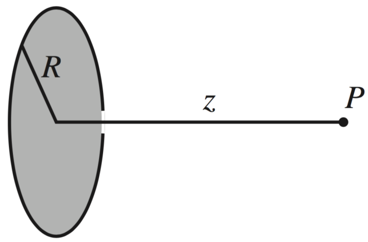

Figure 2: Sketch of a charged disk.

Exercise 3.
(Charged Disk) (Category II)
We consider a disk of radius R, carrying a total charge Q, uniformly spread over the disk (Fig. 2).

a) Calculate the electric field at a distance z along the disk's axis.

b) By considering the $R \to \infty$ limit, find the electric field generated by a charged infinite plane. Hint: For a charged infinite plane, the surface charge density σ can be assumed finite.

c) When R is finite, discuss the $z \gg R$ and $z \ll R$ limits. Hint: consider the electric field in the given limiting cases up to leading order.

Exercise 4.
(Non-uniformly charged sphere/Category II)

An insulating solid sphere with radius R is unevenly charged. The positive charge density is described by $\rho = Kr$ where K is a positive constant (with units of C/m^4) and r is the distance (in units of m) from center of the sphere.

a) Assume the sphere to be surrounded by vacuum (Fig. 3). Find the equation describing the magnitude of the electric field E at a distance r from the center of the sphere in terms of constant K and radius R. Consider both cases of $r < R$ and $r > R$. (Hint: the spherical symmetry allows for Gauss's law with appropriately chosen Gaussian surfaces.)

b) Sketch the result $E(r)$ as a function of r from $r = 0$ to $r > R$.

c) Determine the equation for the electric potential function $\phi(r)$ as a function of r in terms of constant K and radius R. Sketch the result and provide the solutions for $\phi(r)$ at $r = 0$ and $r = R$.

d) A point-like negative test charge q with mass m is positioned at a distance $r = 4R$ and first held at rest. Then it is released. Find the equation for the velocity v at $r = 2R$ in terms of K, R, q, and m.

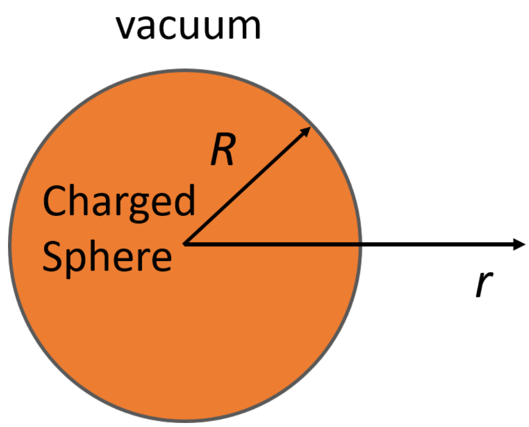

Figure 3: Sketch of the charged sphere with vacuum surrounding.

Solution 1.

We refer to Fig. 4. All dq have the same distance $|\vec{r}-\vec{r}'|$ from the considered point in space. The vector $(\vec{r}-\vec{r}')$ has always the same angle θ with the z-axis. We use the notation $\vec{r}=(x,y,z)$. From Fig. 4 it is clear that $|\vec{r}-\vec{r}'|=a$.

a) We start with the general formula for the potential as stated in the problem: $\phi(\vec{r})=\int \frac{\rho(\vec{r}')dV}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}$. Here, the term ρdV can be identified with an infinitesimal charge dq. The potential can then be written in terms of dq as $\phi=\int \frac{dq}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}$. Since we are considering a 1D line, the amount of charge dq is λdl, where λ is the linear charge density. The potential is then found by integrating over the line path Γ as:
$\phi=\int_\Gamma d\phi=\int_\Gamma \frac{dq}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}=\int_\Gamma \frac{\lambda dl}{4\pi\epsilon_0 a}=\int_0^{2\pi}\frac{\lambda R d\varphi}{4\pi\epsilon_0 a}=\frac{\lambda R}{4\pi\epsilon_0 a}\int_0^{2\pi}d\varphi=\frac{\lambda R}{4\pi\epsilon_0 a}2\pi.$

We conclude that $\phi=\frac{\lambda R}{2\epsilon_0 a}$. To obtain the potential as a function of z one considers $a=\sqrt{z^2+R^2}$. Then $\phi=\phi(z)=\frac{\lambda R}{2\epsilon_0\sqrt{z^2+R^2}}$.

For $z\gg R$ the potential is approximated by $\phi(z)\approx\frac{\lambda R}{2\epsilon_0 z}$. This is the same functional form like the potential of a point-like charge.

b) The electric field is $\vec{E}=-\vec{\nabla}\phi$. Therefore we compute $\vec{\nabla}\phi$.

$\vec{\nabla}\phi=(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z})$. Because of rotational symmetry, the electric field must be along the z-axis. Hence, $\frac{\partial\phi}{\partial x}=0$ and $\frac{\partial\phi}{\partial y}=0$.

We have $\frac{\partial\phi}{\partial z}=\frac{\lambda R}{2\epsilon_0}\left(-\frac{1/2}{(z^2+R^2)^{3/2}}2z\right)=-\frac{z\lambda R}{2\epsilon_0(z^2+R^2)^{3/2}}$.

We conclude that $\vec{E}=-\vec{\nabla}\phi=\left(0,0,\frac{z\lambda R}{2\epsilon_0(z^2+R^2)^{3/2}}\right)$.

For $z\gg R$ the electric field is approximated as follows: $\vec{E}=\hat{z}\frac{z\lambda R}{2\epsilon_0(z^2+R^2)^{3/2}}\approx\hat{z}\frac{\lambda R}{2\epsilon_0 z^2}$. The dependence on the z coordinate is z^{-2} which is like that of a point-like charge.

Solution 2.

We refer to Fig. 5.

a) $[E]=\mathrm{V/m}$ and $[r^3]=\mathrm{m}^3$. The unit vector is without a dimension, i.e. $[\hat{r}]=1$. This leads to $[k]=[E]/[r^3]=\mathrm{V/m}^4$.

b) $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0\rightarrow\rho=\epsilon_0\vec{\nabla}\cdot\vec{E}$. The electric field has radial symmetry. This suggests to use spherical coordinates. Because of radial symmetry of $\vec{E}$, $E_\theta=0=E_\varphi$, and only the radial component of the electric field is non-zero $E_r=kr^3$.

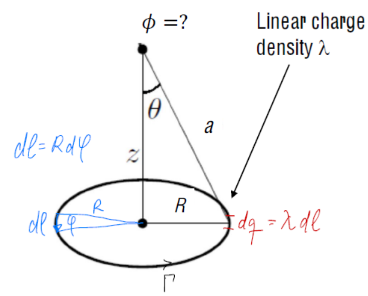

Figure 4: Sketch of the problem. Relevant parameters and infinitesimal quantities used to calculate integrals are defined.

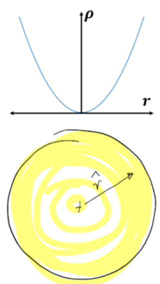

Figure 5: Sketch of problem. Charge density is qualitatively sketched as function of radial distance.

The divergence operator is used in spherical coordinates (Mathematical Tool Box I). Because $\vec{E}$ depends only on r, only the relevant term in the divergence operator is used and $\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{k}{r^2}\frac{\partial r^5}{\partial r} = \frac{5kr^4}{r^2} = 5kr^2$. We conclude that $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = 5\epsilon_0 k r^2 = \rho(r)$. The charge density increases quadratically from the center of the charged body.

c) Charge density as function of radial distance is illustrated in Fig. 5.

d) First we find the value of k: $k = E/r^3 = \frac{5\ \mathrm{kV/m}}{1\ \mathrm{cm}^3} = 5 \cdot 10^9 \frac{\mathrm{V}}{\mathrm{m}^4}$. Then we compute:
$\rho(r = 1\ \mathrm{cm}) = 5\epsilon_0 k r^2 = 5 \cdot (8.85 \cdot 10^{-12}) \cdot 5 \cdot 10^9 \cdot 10^{-4} \frac{\mathrm{As}}{\mathrm{m}^3} = 2.2125 \cdot 10^{-5} \frac{\mathrm{C}}{\mathrm{m}^3} = 1.381 \cdot 10^{14} |e|/\mathrm{m}^3$.

Solution 3.

a) The system is symmetric with respect to z-axis, see Fig. 6. Cylindrical coordinates are used for the following solution.

Exploiting the symmetry of the problem we can argue that all electric field components not aligned with the z-axis cancel out.

The non-zero electric field component is $E_z = \vec{E} \cdot \hat{\mathbf{z}} = E \cdot (\cos\theta) = E \cdot \frac{z}{d}$
$\cos\theta = \frac{z}{d}$ is found from Fig. 6. In addition one finds $d = \sqrt{r^2 + z^2}$.

We consider a uniform surface charge density $\sigma = Q/S$, with S being the total area of the disk. Each element dq on the ring-like area (black) can be evaluated from: $dq = \sigma dr r d\varphi$ where φ is the angle in the plane of the disk. Similar to exercise 1, for each ring-like area, the total charge $dq_{ring} = \int_0^{2\pi} dq = \int_0^{2\pi} \sigma dr r d\varphi = 2\pi\sigma r dr$ The contribution to the electric field amounts to $|d\vec{E}| = \frac{dq}{4\pi\epsilon_0 d^2}$. If we go around such a ring the contributions in x and y directions cancel out and only the components in z direction add up ($d\vec{E}$ needs to be projected on the z axis considering $\cos\theta$). Hence for a ring-like area (black) where dq has a distance d from P, the

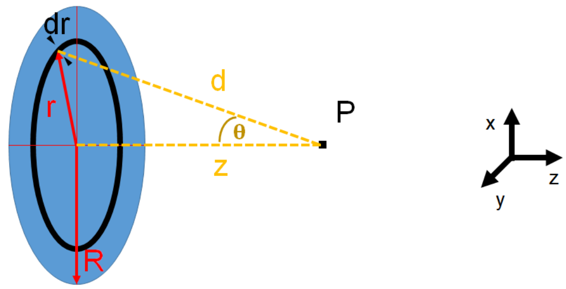

Figure 6: dr is the infinitesimal width of the circular annulus at a distant r from the center of the disk. z is the distance along z-axis of the generic point P from the center of the disk. $d = \sqrt{z^2 + r^2}$ is the distance between a point of the ring (black) to the point P. The point P lies on the z-axis. Because of the symmetry of the problem, off-z-axis components of the electric field cancel out.

electric field contribution at point P is evaluated as follows: $dE_z = \frac{dq_{\text{ring}} \cos\theta}{4\pi\epsilon_0 d^2} = \frac{2\pi\sigma \cos\theta r dr}{4\pi\epsilon_0 (z^2+r^2)}$. We express $\cos\theta$ as follows: $\cos\theta = \frac{z}{\sqrt{z^2+r^2}}$ (see above).

To obtain the total field we integrate over the disk considering a radius R: $E_{\text{tot}}(z) = E_z(z) = \int_0^R \frac{2\pi\sigma z r dr}{4\pi\epsilon_0 (z^2+r^2)^{3/2}} = \frac{\sigma z}{2\epsilon_0} \frac{-1}{\sqrt{z^2+r^2}}\Big|_0^R = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$.

b) If $R \to +\infty$ the electric field found in (a) approaches $E = \frac{\sigma}{2\epsilon_0}$. This agrees with the result obtained for the electric field on one side of an infinitely wide charged plate.

c) For finite R let us look at two limiting cases.

$z \gg R$. Let $x = \frac{R}{z}$. We have to consider the case where $x \ll 1$:

$$E_{\text{tot}}(z) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{\sqrt{1+(R/z)^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{\sqrt{1+x^2}}\right).$$

Performing a Taylor series around $x \to 0$ until leading order gives:

$$E_{\text{tot}}(z) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{\sqrt{1+x^2}}\right) \approx \frac{\sigma}{2\epsilon_0}\left(1 - \left(1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)\right)\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \left(1 - \frac{R^2}{2z^2} + \mathcal{O}(\frac{R^4}{z^4})\right)\right).$$

Thus, the electric field in the limit $z \gg R$ is given by $E \approx \frac{Q}{4\pi\epsilon_0 z^2}$; i.e. the electric field of a point-charge.

$z \ll R$. Let $\chi = \frac{z}{R}$. We have to consider the case where $\chi \ll 1$:

$$E_{\text{tot}}(z) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{\chi}{\sqrt{1+\chi^2}}\right) \approx \frac{\sigma}{2\epsilon_0}\left(1 - 0 + \mathcal{O}(\chi)\right).$$

The electric field in the limit $z \ll R$ is given by $E \approx \frac{\sigma}{2\epsilon_0}$. In the *near field*, the solution looks like the result obtained for the electric field on one side of an infinitely wide charged plate, i.e. a plate capacitor. Capacitors will be discussed in more detail during the next lecture.

Solution 4.

a) For $r \leq R$ from Gauss's law with a Gaussian surface at $r < R$ we have $\oiint \vec{E}.d\vec{a} = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$ with

$$Q_{enc} = \iiint_V \rho\, dV = \int K r' 4\pi r'^2 dr' = 4\pi K \int_0^r r'^3 dr' = 4\pi K \left[\frac{1}{4} r'^4\right]_0^r = \pi K r^4.$$

Therefore, the electric field is obtained as

$$4\pi r^2 E = \frac{1}{\varepsilon_0}\pi K r^4 \qquad \Rightarrow \qquad E(r) = \frac{Kr^2}{4\varepsilon_0}.$$

For $r > R$ again applying Gauss's law $\oiint \vec{E}.d\vec{a} = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$

$$Q_{enc} = \iiint_V \rho dV = \int Kr' 4\pi r'^2 dr' = 4\pi K \int_0^r r'^3 dr' = 4\pi K \int_0^R r'^3 dr' + 4\pi K \int_R^r 0 \times dr' = 4\pi K \left[\frac{1}{4} r'^4\right]_0^R = \pi K R^4.$$

$$4\pi r^2 E = \frac{1}{\varepsilon_0}\pi K R^4 \qquad \Rightarrow \qquad E(r) = \frac{KR^4}{4\varepsilon_0 r^2}.$$

b) The electric field of part (b) is sketched in Fig. 7.

c) To calculate electric potential we have $\phi(r) - \phi(\infty) = -\int_\infty^r E(r')dr'$. We set $\phi(\infty) = 0$. For $r \geq R$

$$\phi(r) = -\int_\infty^r \frac{KR^4}{4\varepsilon_0 r'^2}\, dr' = \left[\frac{KR^4}{4\varepsilon_0 r'}\right]_\infty^r = \frac{KR^4}{4\varepsilon_0 r}.$$

For $r < R$

$$\phi(r) - \phi(\infty) = -\int_\infty^r E(r')dr' = -\int_\infty^R E(r')dr' - \int_R^r E(r')dr'$$

$$\phi(r) = \phi(R) - \int_R^r E(r')dr' = \phi(R) - \int_R^r \frac{Kr'^2}{4\varepsilon_0} dr' = \phi(R) - \left[\frac{Kr'^3}{12\varepsilon_0}\right]_R^r$$

$$\phi(r) = \frac{KR^3}{4\varepsilon_0} - \left(\frac{Kr^3}{12\varepsilon_0} - \frac{KR^3}{12\varepsilon_0}\right) = \frac{KR^3}{12\varepsilon_0}\left(4 - \frac{r^3}{R^3}\right)$$

Sketch of the potential is given in Fig. 8.

d) According to energy conservation law, the electric potential energy ($E^P = q\phi(r)$) of the charged particle will change into kinetic energy ($E^K = \frac{1}{2}mv^2$)

$$E_1^P + E_1^K = E_2^P + E_2^K$$

$$E_1^P = E^P(4R) = -|q|\frac{KR^4}{4\varepsilon_0(4R)}; \qquad E_2^P = E^P(2R) = -|q|\frac{KR^4}{4\varepsilon_0(2R)}$$

$$E_1^K = E^K(4R) = 0; \qquad E_2^K = E^K(2R) = \frac{1}{2}mv^2$$

$$-|q|\frac{KR^3}{16\varepsilon_0} = -|q|\frac{KR^3}{8\varepsilon_0} + \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = |q|\frac{KR^3}{4\varepsilon_0}\left(\frac{1}{2} - \frac{1}{4}\right) = |q|\frac{KR^3}{16\varepsilon_0}$$

$$\vec{v} = \sqrt{\frac{|q|}{m}\frac{KR^3}{8\varepsilon_0}}\,\hat{e}_r.$$

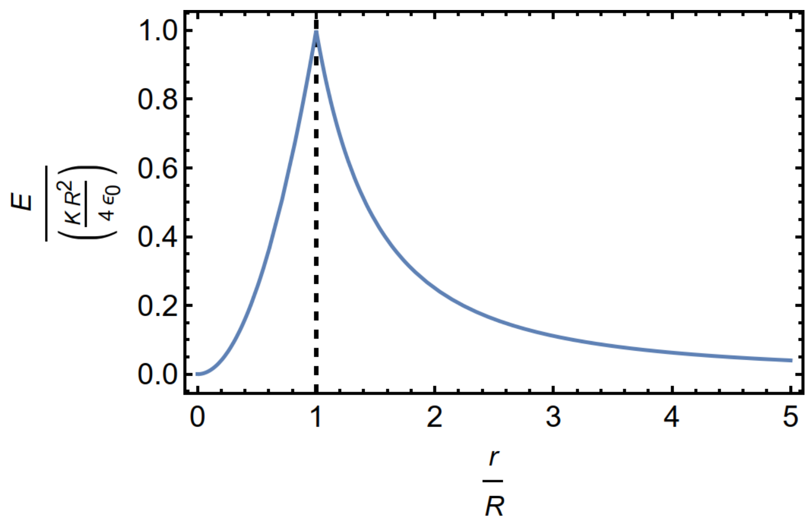

Figure 7: Sketch of the electric field E as a function of radial distance r.

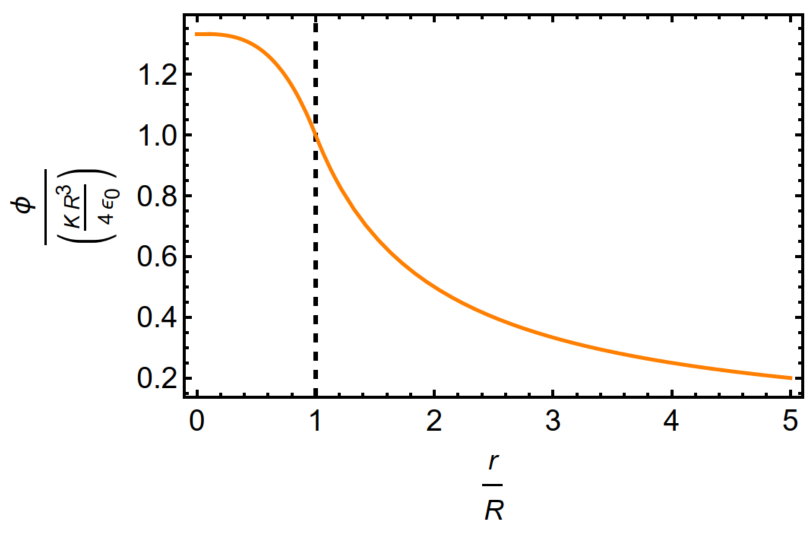

Figure 8: Sketch of the electric potential ϕ as a function of radial distance r.