Exercise sheet 2: Electric Fields

18/09/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). Problems addressing pure mathematical aspects are extra and not rated. For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions).The aim here is to show how in an exam context one should approach the problem making use of symmetry analysis and explaining the development of the entire solution step-by-step. The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Electric fields from opposite charges put on a straight line)(Category I)

We consider two charges q of opposite sign, but with the same magnitude, to reside on the x axis. The separation between them is d.

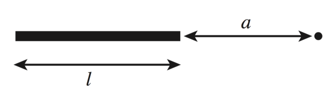
a) Find the electric field, $\vec{E}$ (magnitude and direction) a distance z above the midpoint between them, as a function of z and d.

b) Which dependence of $\vec{E}$ on z is predicted for $z \gg d$? Compare with the formula describing the electric field of a dipole moment that you might search for in a textbook. What do you conclude?

Hint: Consider the decomposition of vectors along x- and y- axis as done in the Exercise 2 of Exercise sheet 1.

Exercise 2.
(Charged Rod) (Category I)

We consider a rod of small radius such that it can be approximated as a line with length l, carrying a uniformly distributed charge Q.



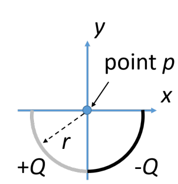
a) Define the linear charge density λ.

b) Find the electric field at a distance a from the end of the rod, along its axis (see the sketch). The electric field can be calculated with the following equation:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_\Gamma \frac{\lambda(\vec{r}\prime)dl}{|\vec{r} - \vec{r}\prime|^3} (\vec{r} - \vec{r}\prime) \tag{1}$$

c) How does the calculated field behave for $a \gg l$ and for $a \ll l$? Hint: consider the electric field in the given limiting cases up to leading order.

Exercise 3.
(Charged semi-circle/taken from an exam) (Category II)
An ultra-thin rod is bent to a semi-circle and contains distributed charges as sketched (see figure). The charges $+Q$ and $-Q$ are each homogeneously distributed in the indicated segments of the semi-circle. Calculate the formula for the electric field vector $\vec{E}$ in point P as a function of $|Q|$ and the radius r. (For the calculation assume the rod to be a line mathematically)



Solution 1.

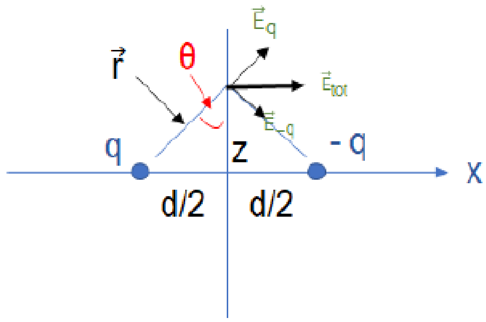


Figure 1: Sketch of the problem with reference system and relevant specified parameters. The electric field vectors of the charges sum up and by superposition provide the total electric field. The electric field components along z cancel each other.

a) We consider Fig. 1
By superposition principle $\vec{E}_{tot} = \vec{E}_{-q} + \vec{E}_q$.
Given the geometry of the system and the opposite charges we notice that $E_{q,z} = -E_{-q,z}$ and $E_{q,x} = E_{-q,x}$.
We define $E = E_{q,x}$. The total electric field is then written as $\vec{E}_{tot} = 2E\hat{x}$. We need to compute E.
$E = E_{q,x} = \sin\theta \frac{q}{4\pi\epsilon_0 r^2}$, with $r^2 = z^2 + d^2/4$. Using the definition of sine function we replace $\sin\theta = \frac{d/2}{r}$.
Combining all together we find: $E = E_{q,x} = \sin\theta \frac{q}{4\pi\epsilon_0 r^2} = \frac{d/2}{r} \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} = \frac{q}{4\pi\epsilon_0} \frac{d/2}{r^3} = \frac{q}{4\pi\epsilon_0} \frac{d/2}{(z^2+d^2/4)^{3/2}}$. The total field is: $\vec{E}_{\text{tot}} = 2E\hat{x} = \frac{q}{4\pi\epsilon_0} \frac{d}{(z^2+d^2/4)^{3/2}}\hat{x}$.

b) For z $\gg$ d the expression of the total electric field is approximated by $\vec{E}_{\text{tot}} = \frac{q}{4\pi\epsilon_0} \frac{d}{(z^2+d^2/4)^{3/2}}\hat{x} = \frac{q}{4\pi\epsilon_0} \frac{d}{z^3(1+(d/z)^2/4)^{3/2}}\hat{x} \approx \frac{q}{4\pi\epsilon_0} \frac{d}{z^3}\hat{x}$. This formula reproduces the electric field generated by a dipole moment, i.e., two opposite charges of magnitude q arranged on a straight line at distance d. Such electric dipole moments play a role in Chapter 2 of the course.

Solution 2.

a) This represents a 1D problem. Charge density λ is thereby given per unit of length: $\lambda = \frac{Q}{L}$

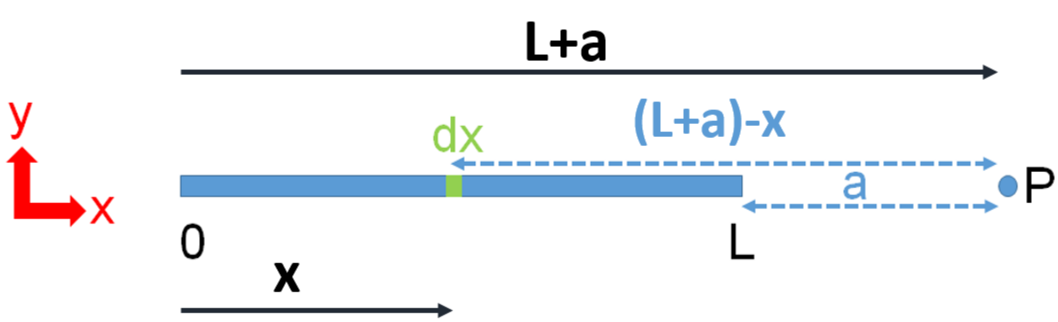


Figure 2: R is the distance from the arbitrary point x within the wire with infinitesimal width dx.

b) We know the electric field for a line charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_\Gamma \frac{\lambda(\vec{r\prime})dl}{|\vec{r}-\vec{r\prime}|^3}(\vec{r}-\vec{r\prime}) \tag{2}$$

It is important to note that the unit vector $\frac{\vec{r}-\vec{r\prime}}{|\vec{r}-\vec{r\prime}|}$ is not constant and cannot be taken out of the integral. Since in this problem, we have a system that is symmetric with respect to spatial inversion along y-axis, we concern ourselves with only the $\hat{x}$ direction.

Let us take an infinitesimally small component dx at a distance x along the charged rod, then $dl = dx$ and the infinitesimally small charge dq is $\lambda(x)dx$. So,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda(x)dx}{(|\vec{r}-\vec{r\prime}|)^2}\hat{x} \tag{3}$$

In our symmetry considerations, $(\vec{r}-\vec{r\prime}) = ((L+a)-x)\hat{x}$, $|\vec{r}-\vec{r\prime}| = ((L+a)-x)$. Since, the linear charge density is constant ($\lambda(x) = \lambda$), it can be taken out of the integral, and using $(\vec{r}-\vec{r\prime}) = ((L+a)-x)$ we arrive at the solution:

$$\begin{aligned}\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{dx}{((L+a)-x)^2}\hat{x} &= \frac{\lambda}{4\pi\epsilon_0}\left.\frac{1}{((L+a)-x)}\right|_0^L \hat{x} \\ &= \frac{\lambda}{4\pi\epsilon_0}(\frac{1}{a}-\frac{1}{L+a})\hat{x}\end{aligned} \tag{4}$$

c) Let us analyse the two limiting cases:

$a \gg L$. Let $x = \frac{L}{a}$. We have to consider the case where $x \ll 1$:

$$\vec{E}(x_p) = \frac{\lambda}{4\pi\epsilon_0}(\frac{1}{a}-\frac{1}{L+a})\hat{\mathbf{x}} = \frac{\lambda}{4\pi\epsilon_0}\frac{L}{a^2\left(1+\frac{L}{a}\right)}\hat{\mathbf{x}} = \frac{\lambda}{4\pi\epsilon_0}\frac{x}{a\,(1+x)}\hat{\mathbf{x}}.$$

Performing a Taylor series around $x \to 0$ until leading order gives:

$$\vec{E}(x_p) \approx \frac{\lambda}{4\pi\epsilon_0}\frac{1}{a}\left(x+\mathcal{O}(x^2)\right)\hat{\mathbf{x}}.$$

Thus the electric field is approximated by $\vec{E}(x_p) \approx \frac{1}{4\pi\epsilon_0}\lambda\frac{L}{a^2}\hat{\mathbf{x}}$ This reflects the Coulomb law for a point charge with $Q = \lambda L$.

$a \ll L$. Let $\chi = \frac{a}{L}$. We have to consider the case where $\chi \ll 1$:

$$\vec{E}(x_p) = \frac{\lambda}{4\pi\epsilon_0}(\frac{1}{a}-\frac{1}{L+a})\hat{\mathbf{x}} = \frac{\lambda}{4\pi\epsilon_0}\frac{L}{La\left(1+\frac{a}{L}\right)}\hat{\mathbf{x}} = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{a\,(1+\chi)}\hat{\mathbf{x}}.$$

Performing a Taylor series around $\chi \to 0$ until leading order gives:

$$\vec{E}(x_p) \approx \frac{\lambda}{4\pi\epsilon_0}\frac{1}{a}\left(1+\mathcal{O}(\chi)\right)\hat{\mathbf{x}}.$$

The electric field is thus approximated by: $\vec{E}(x_p) = \frac{1}{4\pi\epsilon_0}\lambda\frac{1}{a}\hat{\mathbf{x}}$. Hence we find a distance dependence $E \propto \lambda/a$.

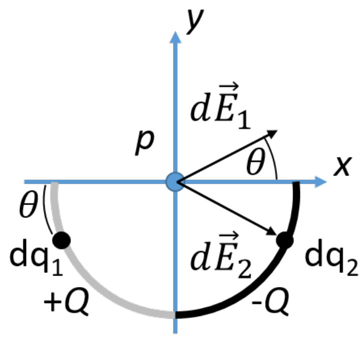


Figure 3: Sketch of the electric fields $d\vec{E}_1$ and $d\vec{E}_2$ generated by dq_1 and dq_2, respectively.

Solution 3.

Note: this problem was added for special training on written-exam level problems. (Hint for checking your preparation for the written exam: in the exam a duration of about 20 min is estimated for the correct solution which includes a literal description (outline) of the symmetry analysis.)

Symmetry analysis

We consider the field generated by the charge $+Q$ and that of charge $-Q$, as shown in Fig. 3.

$d\vec{E}_1$ is the field contribution generated by dq_1, an infinitesimal portion of the $+Q$ charge. We also consider $d\vec{E}_2$, the field contribution generated by an infinitesimal portion dq_2 of charge $-Q$, residing at a symmetric point with respect to the y-axis.

We know that $dq_1 = -dq_2$, i.e. $dq_1 = +|dq|$, $dq_2 = -|dq|$. We look at the total field at point p and we consider the superposition principle. By the superposition principle, $\vec{E}_{tot}(p) = \vec{E}_1(p) + \vec{E}_2(p)$, where 1 (2) stands for $+Q$ ($-Q$).

$\vec{E}_{tot}(p)$ is the vectorial sum of all electric fields at the point p. From Fig. 3, we deduce that $dE_{1,x} = dE_{2,x}$ and $dE_{1,y} = -dE_{2,y}$. Summing all contributions, we notice that all y-components cancel out. Therefore: $\vec{E}_{tot}(p) = E_{tot}\hat{x} = 2E_{1,x}\hat{x}$. Thus, we conclude that to obtain the total electric field, we need to compute $E_{1,x}$ via integration, that is, the component along x of the electric field generated by the total charge $+Q$, and multiply the result with 2.

Calculation of E_x

To compute $E_{1,x}$, consider

$$dE_{1,x} = dE_1 \cos(\theta), \tag{5}$$

where θ is the angle between the electric field and the x-axis (see Fig.3). The distance from the point p is constant and equal to r. The electric field due to an infinitesimal charge portion dq_1 can then be written as

$$dE_1 = \frac{dq_1}{4\pi\epsilon_0 r^2}. \tag{6}$$

The infinitesimal charge can be related to a portion of the semi-circle via the linear charge density λ: $dq = \lambda r d\theta$. Substituting this into Eq. 6, and substituting the resulting expression in Eq. 5 gives:

$$dE_{1,x} = \frac{\lambda r}{4\pi\epsilon_0 r^2}\cos(\theta)\,d\theta = \frac{\lambda}{4\pi\epsilon_0 r}\cos(\theta)\,d\theta. \tag{7}$$

Since $+Q$ is uniformly distributed, λ is constant and can be put in front of the integral.
From this, we compute $E_{1,x}$ by integrating over the angle θ between 0 and $\pi/2$:

$$E_{1,x} = \int dE_x = \frac{\lambda}{4\pi\epsilon_0 r}\int_0^{\pi/2}\cos(\theta)\,d\theta = \frac{\lambda}{4\pi\epsilon_0 r}\sin(\theta)|_0^{\pi/2} = \frac{\lambda}{4\pi\epsilon_0 r} \tag{8}$$

We consider $\lambda = \frac{Q}{\frac{2\pi r}{4}} = \frac{2Q}{\pi r}$ and replace λ accordingly:

$$E_{1,x} = \frac{2Q/\pi r}{4\pi\epsilon_0 r} = \frac{Q}{2\pi^2\epsilon_0 r^2}. \tag{9}$$

Finally, using the previously derived relation between $\vec{E}_{tot}$ and $E_{1,x}$, we find for $\vec{E}_{tot}$:

$$\vec{E}_{tot}(p) = 2E_{1,x}\hat{x} = 2\frac{Q}{2\pi^2\epsilon_0 r^2}\hat{x} = \frac{Q}{\pi^2\epsilon_0 r^2}\hat{x}. \tag{10}$$