Exercise sheet 1: Vector Algebra Tools

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11/09/2024

By this problem sheet we intend to *stimulate preliminary considerations* about concepts related to vector algbra, which are relevant for the course. This exercise acts as a revision of the basic concepts.

Exercise 1.

Consider that a hockey puck with a mass of 0.5 kg slides on a friction-less horizontal surface of an ice rink. Two sticks strike the puck simultaneously as shown in figure 1. The first stick exerts a force $\vec{F_1}$ with a magnitude of 4.0 N, directed at an angle of -60° to the x axis. The second stick exerts a force $\vec{F_2}$ with a magnitude of 5.0 N at an angle of +20° to the x axis. Determine both the magnitude and the direction of the puck's acceleration.

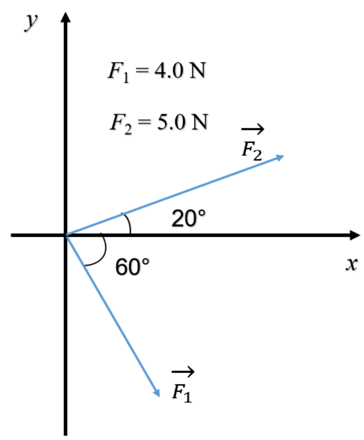

Figure 1: A hockey puck experiencing two forces

Exercise 2.

Consider that a car is on a friction-less inclined plane at an angle of θ as shown in Figure 2.

Part A. Draw a diagram showing the forces acting on the car and calculate the acceleration of the car. Discuss what happens if the inclination θ is 0°, 30°, 45°, and 90°.

Part B. How long does it take for the front bumper of the car to reach the bottom of the inclined plane, and what is the car's velocity when it reaches there.? Assume that the distance between the front bumper of the car to the bottom of the inclined plane is d.

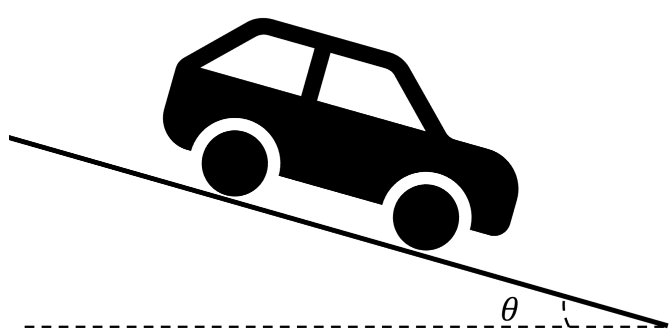

Figure 2: A car on a friction-less inclined plane

Exercise 3.
A space transportation vehicle releases a communication satellite with a mass $m = 500\,\text{kg}$ while in an orbit with a height $r_i = 300\,\text{km}$ above the surface of the Earth, as shown in Figure 3. The radius of the Earth is $R_E = 6370\,\text{km}$ and the mass M of the Earth is 5.97×10^{24}kg. **Hint:** The gravitational force reads $\vec{F} = -\frac{GMm}{r^2}\vec{e_r}$ where $\vec{e_r}$ is

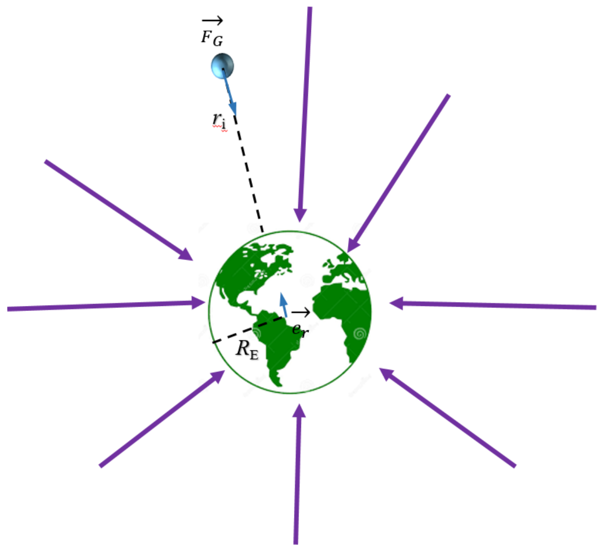

Figure 3: The satellite boosted from r_i to r_f

the unit vector pointing away from the center of the Earth and r is the distance from the center of the Earth, the satellite has a circular orbit.
Part A. How much is the kinetic energy of the satellite at this height?
Part B. What is the total mechanical energy at this height? Hint: Calculate the potential energy at a distance r ($r > R_E$) from the center of the earth. The mechanical energy is the sum of the kinetic and potential energy.
Part C. A rocket engine on the satellite is required to boost it into a circular geosynchronous orbit at 36 000 km. How much energy does the engine have to provide?

1 Hints and Examples

We start with some examples on the basic vector algebra, which will help solve the problems given in the exercise. Consider figures 4 to 7 given below:

1. Consider the case of collinear forces: Finding the net force of the two forces shown in figure 4, $\vec{F_1}$ and $\vec{F_2}$.

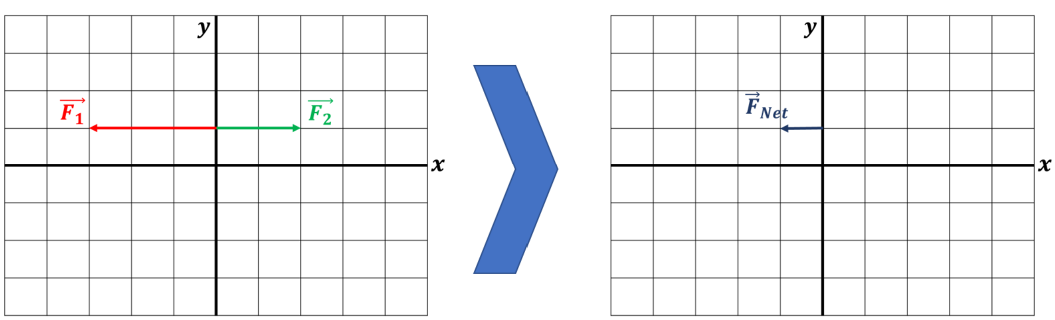

Figure 4: Net of two forces collinear with each other

2. Consider the case of non-collinear forces: Finding the net force of the two forces shown in figure 5, $\vec{F_1}$ and $\vec{F_2}$. This can be done in two ways: (i) Using the graphical *tip-to-tail* method, or (ii) by decomposing the vectors in their components on the x- and y-axis. Note that the choice of axis is important and relevant in Exercise 2 of the problem sheet.

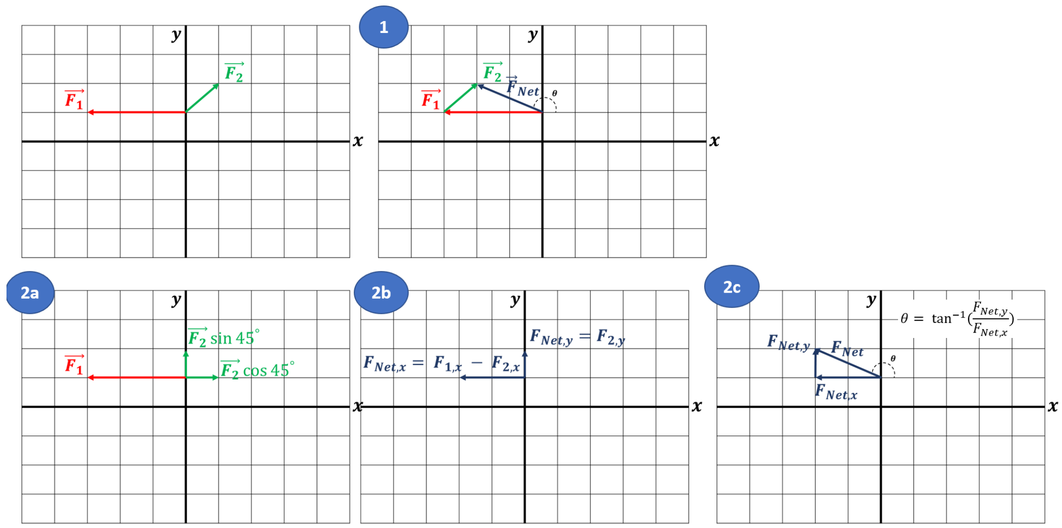

Figure 5: Net of two forces acting on a body. The first method marked (1) is the *tip-to-tail* method, and the second method marked (2a-c) is solving by resolving a vector on the axes.

3. Consider the case of more than two non-collinear forces: Finding the net force of the two forces shown in figure 6, $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$. *Tip-to-tail* method is shown here.

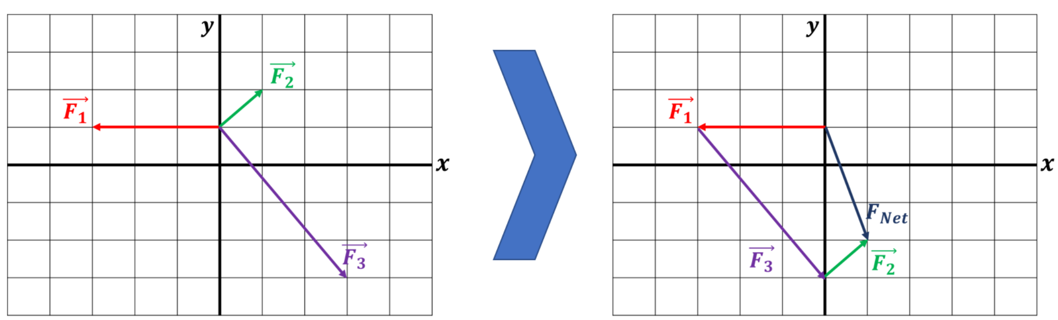

Figure 6: Net of three forces acting on a body

4. Consider taking components in a three-dimensional space as shown in figure 7. Note that this example helps in moving from Cartesian to Spherical coordinates.

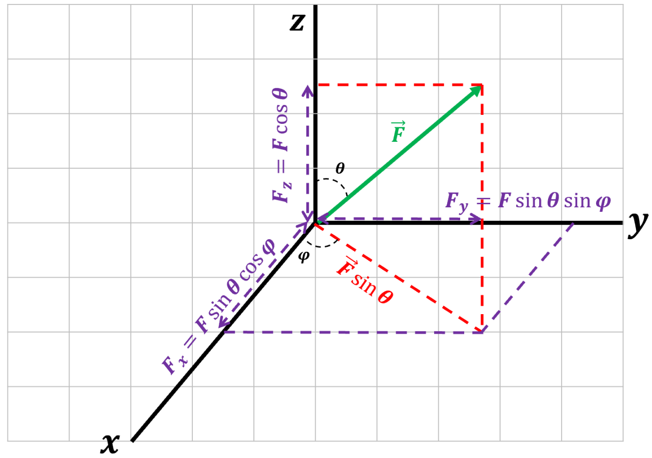

Figure 7: Example of a vector in three-dimensional space | Resolving the given vector in components projected on the x,y,z axes

Solution 1.

We model the puck as accelerating due to the effect of the two forces due to the sticks only, and then apply Newton's second law. And the diagram is shown in Figure 8

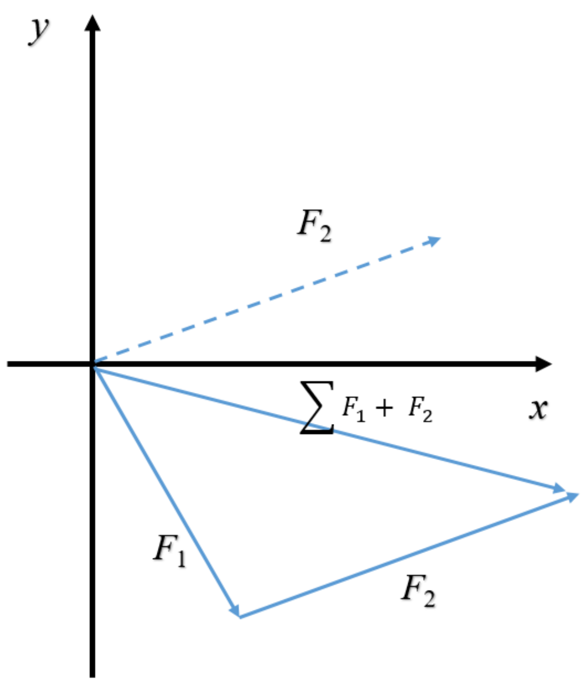

Figure 8: The vector addition of the forces $\vec{F_1}$ and $\vec{F_2}$ to give the resultant net force

Find the component of the net force acting on the puck in the x direction:

$$\begin{aligned} \sum F_x &= F_{1x} + F_{2x} \\ &= F_1 \cos(-60^\circ) + F_2 \cos 20^\circ \\ &= (4.0N)(0.500) + (5.0N)(0.940) \\ &= 6.7 \text{ N} \end{aligned} \tag{1}$$

Find the component of the net force acting on the puck in the y direction:

$$\begin{aligned} \sum F_y &= F_{1y} + F_{2y} \\ &= F_1 \sin(-60^\circ) + F_2 \sin 20^\circ \\ &= (4.0N)(-0.866) + (5.0N)(0.342) \\ &= -1.8 \text{ N} \end{aligned} \tag{2}$$

Use Newton's second law in component form to find the x and the y components of the puck's accelerations:

$$a_x = \frac{\sum F_x}{m} = \frac{6.7N}{0.50kg} = 13.4 \text{ m/s}^2 \tag{3}$$

$$a_y = \frac{\sum F_y}{m} = \frac{-1.8N}{0.50kg} = -3.6 \text{ m/s}^2 \tag{4}$$

Find the magnitude of the acceleration:

$$a = \sqrt{(13.4m/s^2)^2 + (-3.6m/s^2)^2} = 14 \text{ m/s}^2 \tag{5}$$

Find the direction of the acceleration relative to the positive x axis:

$$\theta = \arctan\left(\frac{a_y}{a_x}\right) = \arctan\left(\frac{-3.6}{13.4}\right) = -15^\circ \tag{6}$$

Solution 2.

Part A. Since the inclined plane is friction-less, only two forces act on the car - (i) the gravitational force, and (ii) the Normal force exerted by the inclined plane $\vec{n}$. Figure 9(a) shows the two forces on the car. You can choose to resolve the forces (preferably) parallel to the incline plane (x), and perpendicular to the incline plane (y). Now, since the car can only move in directions parallel to the inclined plane, this would mean that the net force in the direction perpendicular to the incline plane is zero.

$$\sum \vec{n} - \vec{F_{g,y}} = 0 \tag{7}$$

$$\Rightarrow n = F_{g,y} = mg\cos\theta \tag{8}$$

However, the net force in the x-direction might not be zero, so let it be $\vec{F_x}$.

$$\sum \vec{F_x} = mg\sin\theta = ma_x \Rightarrow a_x = g\sin\theta \tag{9}$$

If the inclination is 0, we notice that the acceleration $a_x = g\sin 0^\circ = 0$. This means that there is not net force to accelerate the car in the direction parallel to the inclined plane (not inclined anymore). Similarly, if the inclination is (i) 30°, $a_x = g\sin 30^\circ = \frac{g}{2}$, (ii) 45°, $a_x = g\sin 45^\circ = \frac{g}{\sqrt{2}}$, and (iii) 90°, $a_x = g\sin 90^\circ = g$. We see that the acceleration is independent of the mass of the car, and depends only on the angle of inclination. For no inclination, the car will not move, as the angle of inclination increases, the acceleration of the car increases. At 90°, the car would be in free fall. In the case of $\theta = 90^\circ$, it is also interesting to note that the normal force $n = mg\cos 90^\circ = 0$.

Part B. We can reduce this problem to a simpler problem of a single particle under a constant acceleration of $a_x = g\sin\theta$.

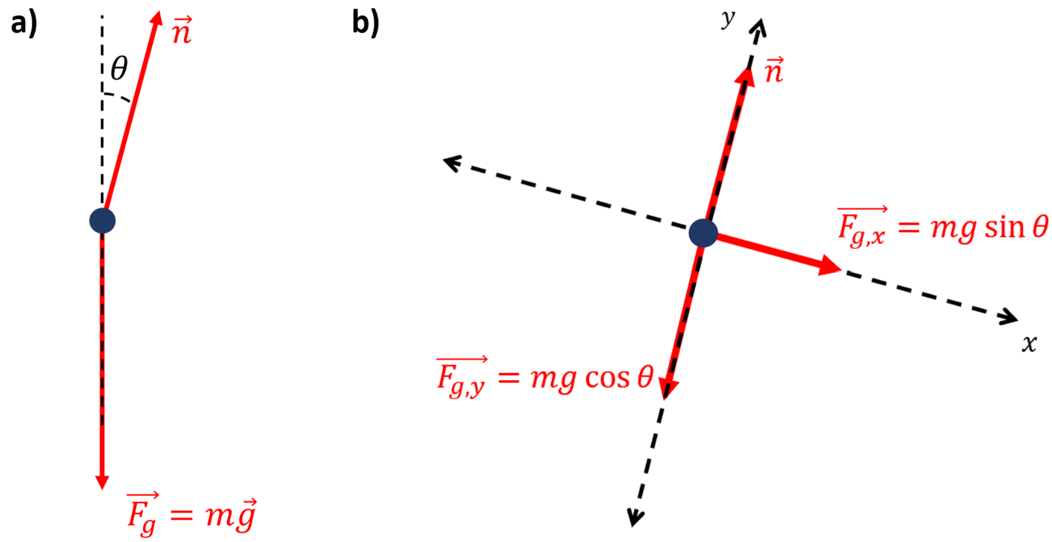

Figure 9: a) The free body diagram of the forces acting on the car. The blue dot represents the centre of mass of the body. b) Resolving the forces parallel and perpendicular to the inclined plane.

Defining the boundaries: the initial position of the front of the bumper is $x_i = 0$, and the final position, i.e. the bottom of the inclined plane is $x_f = d$. The car starts with a zero initial velocity $v_{xi} = 0$. We apply the equations of motions:

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \tag{10}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \tag{11}$$

Using the boundaries in equation 10, and the acceleration $a_x = g\sin\theta$,

$$d = \frac{1}{2}g\sin\theta t^2 \Rightarrow t = \sqrt{\frac{2d}{g\sin\theta}} \tag{12}$$

Using the value of t in equation 11, with the boundaries and the acceleration $a_x = g\sin\theta$, we can find the final velocity of the car as:

$$v_{xf} = \sqrt{2g\sin\theta d} \tag{13}$$

Solution 3.

Part A.

In this case, Newton's law applied to the object of satellite situated at a distance $r = R_E + r_i$ from the centre of Earth is:

$$F_g = ma \Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow \frac{GMm}{r} = mv^2 \Rightarrow \frac{GMm}{2r} = \frac{mv^2}{2} \tag{14}$$

Therefore, we can find the kinetic energy of the satellite,

$$\begin{aligned} K = \frac{mv^2}{2} &= \frac{GMm}{2r} \\ &= \frac{GMm}{2(r_i + R_E)} \\ &= \frac{(6.67\times10^{-11}N\cdot m^2/kg^2)(5.97\times10^{24}kg)(500kg)}{2\times(6.37\times10^6m + 300\times10^3m)} \\ &= 1.5\times10^{10}\text{ J} \end{aligned} \tag{15}$$

Part B. The total mechanical energy E of the system is the sum of the kinetic energy and the potential energy. The change in the potential energy of a system associated with a given displacement of part of the system is defined as $\Delta U = U_f - U_i = -\int_{R_i}^{R_f} F(\vec{r}) d\vec{r}$. The satellite is subject to the gravitational force: $\vec{F} = -\frac{GMm}{r^2}\hat{r}$, therefore, we can find that:

$$\begin{aligned} U_f - U_i &= -\int_{R_i}^{R_f} F(\vec{r}) d\vec{r} = GMm \int_{R_i}^{R_f} \frac{\hat{r}.d\vec{r}}{r^2} \\ &= GMm \int_{R_i}^{R_f} \frac{dr}{r^2} \\ &= -GMm(\frac{1}{R_f} - \frac{1}{R_i}) \end{aligned} \tag{16}$$

For the choice of the reference configuration for the potential energy, it is customary to choose it for zero potential energy to be such that the force they exert on each other is zero. Taking $U_i = 0$ at $R_i = \infty$, then we obtain that:

$$U(r) = -\frac{GMm}{r} \tag{17}$$

We find the total mechanical energy E:

$$\begin{aligned} E &= K + U \\ &= \frac{mv^2}{2} - \frac{GMm}{r} \\ &= -\frac{GMm}{2r} \\ &= -1.5 \times 10^{10} \text{ J} \end{aligned} \tag{18}$$

Part C. Based on the total mechanical energy obtained above, we can find the difference in energies for this satellite-Earth system with the satellite at the initial and final radii:

$$\begin{aligned} \Delta E &= E_f - E_i \\ &= -\frac{GMm}{2 \times (r_f + R_E)} - (-\frac{GMm}{2 \times (r_i + R_E)}) \\ &= -\frac{GMm}{2}(\frac{1}{r_f + R_E} - \frac{1}{r_i + R_E}) \\ &= -\frac{(6.67 \times 10^{-11} N \cdot m^2/kg^2)(5.97 \times 10^{24} kg)(500kg)}{2} \times (\frac{1}{3.6 \times 10^7 m + 6.37 \times 10^6 m}) - (\frac{1}{3 \times 10^5 m + 6.37 \times 10^6 m}) \\ &= 1.25 \times 10^{10} \text{ J} \end{aligned} \tag{19}$$