Exercise sheet 14: Reflection, refraction, interaction of EM waves with matter, Poynting vector, Gauss Law, Magnetic fields, Ampère's Law

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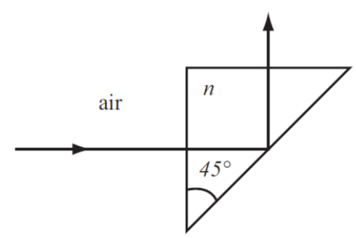
18/12/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(EM wave in an absorbing material (good conductor)/Category I)

Consider an electromagnetic wave (EM) $\vec{E}(\vec{x},t) = E_0\hat{y}\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ propagating along $\vec{x}$-direction in a *lossy* material. We assume metallic aluminium (Al) exhibiting a complex refractive index $\tilde{n} = n + i\kappa$ with $\kappa > 0$. In a lossy material $\vec{k} = \frac{\omega\tilde{n}}{c}\hat{k}$ with $\hat{k}$ being the unit vector in propagation direction. We consider light that in vacuum has a wavelength λ of 546 nm. For this one finds $n = 0.82$ and $\kappa = 5.99$ in Al. Show that the light wave exhibits a decaying amplitude when entering Al under normal incidence and quantify the so-called decay length after which the intensity diminishes to $1/e$. Compare this value with the wavelength.

Exercise 2.
(Refraction and (total internal) reflection/Category II
(After training for solution: 10 min))



In the figure we sketch a situation where no light beam leaves a prism at the right edge (= total internal reflection at the second surface that the light hits). The refractive index n is such that the angle of the refracted beam at the right surface is just 90°. This is the definition of the critical angle θ_c for total internal reflection.

a) How large is n of the prism assuming that there is air outside the prism with a refractive index equal to 1?

b) What kind of material would exhibit such a value n?

c) Where does the incident light beam go when the index n is (i) doubled, and (ii) halved? Calculate the refraction angle when refracted light is expected.

The following four problems review topics of the previous weeks/chapters in a way which is consistent with a written exam problem

Exercise 3.
(Poynting vector concept applied to current in a perfect coaxial cable/Category II (After training for solution: 20 min))

A coaxial cable consists of two concentric long hollow cylinders of zero resistance (perfect conductors); the inner has a radius a, the outer has radius b, and the length of both is l, with $l \gg b$, as shown in Fig. 1. The cable transmits power from a battery to a load via a DC current I. The battery provides an electromotive force ε between the two conductors at one end of the cable, and the load is a resistance R connected between the two perfect conductors at the other end of the cable.

a) We consider that the battery charges the inner conductor to a charge -Q and the outer conductor to a charge +Q due to the potential difference applied to them. At the same time a current I flows down the inner conductor and back up the outer one as power is dissipated in R. Find the direction and magnitude of the electric field $\vec{E}$ everywhere.

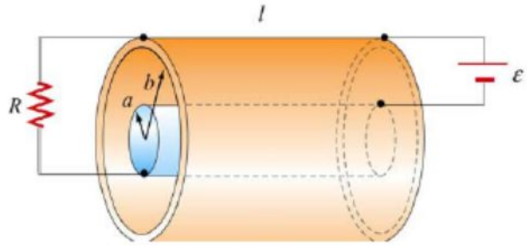


Figure 1: A coaxial cable consists of two concentric long hollow cylinders of zero resistance.

b) Find the direction and magnitude of the magnetic field $\vec{B}$ everywhere.

c) Apply the concept of the Poynting vector $\vec{S}$ inside the coaxial cable and calculate $\vec{S}$.

d) By integrating $\vec{S}$ over an appropriate surface, find the power that flows into the coaxial cable.

e) How does your result in (d) compare to the power dissipated in the resistor?

Exercise 4.
(Loop on string/Category II (taken from an exam) time: 25 min)
Consider a current-carrying wire with a constant current I_1 that is infinitely long. The current is along y-direction. A square loop formed by a massless rigid conductor is positioned symmetrically above the wire as sketched. The loop is parallel to the x, y-plane. The loop carries a constant current I_2 and resides on a rigid string that is at height $z = h$ and parallel with the wire. The loop can slide along and rotate around the nonconducting string without friction. The directions of currents flowing in the closed conductor loop and in the wire are indicated by arrows.

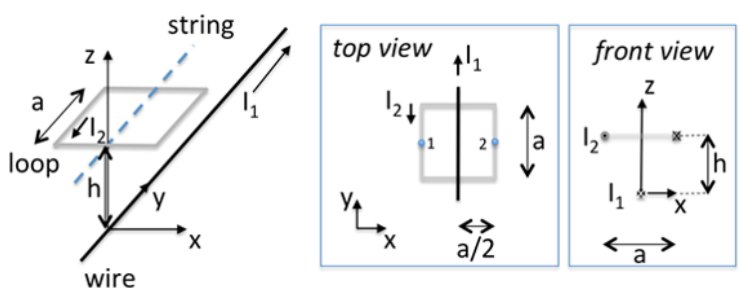


Figure 2: Rectangular loop made of a perfect conductor (grey) positioned on a string (broken line). Separated by a height h a current-carrying wire (black line) is present. The wire is straight and infinitely long. In the center a top view is shown indicating the two positions for analyzing forces and torques. Geometrical parameters and current-flow directions are defined in the central sketch and on the right image which shows a front view.

a) Calculate forces and torques acting at positions 1 and 2 due to current I_1 (see figure) as a function of separation h and side length a.

b) What is the total force on the string?

c) Does the loop rotate? If yes, what is the sense of rotation?

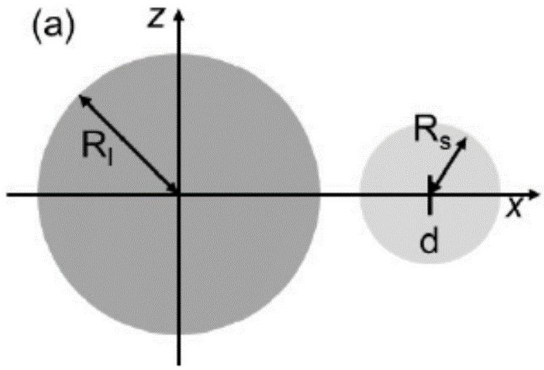


Figure 3: A large solid sphere positioned at the origin of the x, y, z coordinate system, and a small solid sphere located at $x = d$. We show the side view onto the central cross-sectional plane of the charged solid spheres (see text).

Exercise 5.
(Charged Spheres) (Category II time needed after training: 20 min; this is taken from an exam)
Positive electrical charges are uniformly distributed in the volume of a large three-dimensional sphere of radius R_{l}. The value of the uniform charge density is ρ_{C}. Assume that the large sphere of radius R_{l} is positioned next to a small sphere [Fig. 3 (a)]. The central coordinate of the former sphere is at the origin of the coordinate system. The central coordinate of the small sphere is on the x-axis with $x = d$. The small sphere of radius R_{s} is uniformly charged with negative charges of a density $\rho_{\mathrm{s}} = -\rho_{\mathrm{C}}$. The radii of the two spheres are such that they do not touch. Derive the formula for the electrical field vector $\vec{E}$ on the x-axis *inside* the small sphere depending on $\rho_{\mathrm{C}}, R_{\mathrm{l}}, x, \text{and}\, d$.

Exercise 6.
(Optical Fiber) (Category II time needed after training: 15 min; this is taken from an exam)

We consider a cylindrical optical fiber (Fig. 4) consisting of lossless transparent materials A in the core and B in the shell (cladding). The fiber resides in vacuum. The shell of material B surrounds the core. Light of a specific wavelength enters the fiber as sketched in the figure. The materials have different indices of refraction $n_A = 1.480$ and $n_B = 1.440$. All light rays shown are in the same plane which is the mirror plane of the fiber through the central axis.

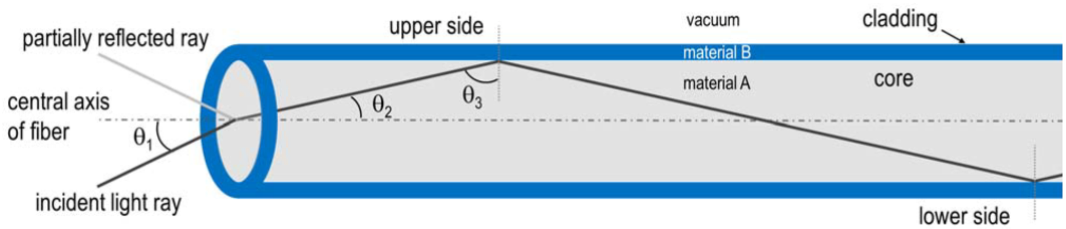


Figure 4: Sketch of the fiber.

a) What is the critical angle θ_3 for the total internal reflection at the interface between materials A and B, i.e., core and cladding?

b) For what range of angle θ_1 is light totally internally reflected at the core-cladding interface?

c) If light is totally internally reflected at the upper core-cladding interface of the fiber, will it be totally internally reflected at the lower core-cladding interface (assuming the relevant interfaces to be parallel)?

Solution 1.

$\tilde{n} = n + i\kappa$ and $\vec{k} = \frac{\omega}{c}\tilde{n}\hat{k} \Rightarrow \vec{k} = (n + i\kappa)\frac{\omega}{c}\hat{k}$

$$\vec{E}(\vec{r},t) = \vec{E}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_0 e^{i(\frac{n\omega}{c}\hat{k}\cdot\vec{r}-\omega t)} \cdot e^{i(i\frac{\kappa\omega}{c}\cdot\hat{k}\cdot\vec{r})} = \underbrace{E_0 e^{i(\frac{n\omega}{c}\hat{k}\cdot\vec{r}-\omega t)}}_{①} \cdot \underbrace{e^{-(\frac{\kappa\omega}{c}\cdot\hat{k}\cdot\vec{r})}}_{②}$$

where ① is the description of an electromagnetic wave in a lossless medium, and ② represents the attenuation, i.e., exponential decay, of the wave.

Assuming, $\hat{k}\cdot\vec{r} = x$, $\Rightarrow \vec{E}(x,t) = E_0 e^{i(\frac{n\omega}{c}x-\omega t)} \cdot e^{-(\frac{\kappa\omega}{c}x)}$ Putting in the values given in the problem,

$\Rightarrow \vec{E}(x,t) = E_0 e^{i(\frac{x}{106.0\text{ nm}}-3.45\times10^{15}t)} \cdot e^{-(\frac{x}{14.5\text{ nm}})}$. To find the decay length we have to consider the intensity $I \propto |\vec{E}|^2$. Using the derived expression of the electric field, we get a decay length of $\lambda_{\text{decay}} = \frac{c}{2\kappa\omega} = 7.3$ nm, which is much smaller than the vacuum wavelength of the incoming light (546 nm).

Solution 2.

a) Using the Snell's law with the notation from the problem set and the sketch in Fig. 5, we obtain

$$\frac{\sin\theta_r}{\sin\theta_i} = \frac{n_I}{n_{II}} = \frac{\sin 90°}{\sin 45°} \Rightarrow n_I = n = \frac{1\cdot 1}{\sin 45°} = \sqrt{2},$$

where n_{II} represents the refractive index of air, and $n_I = n$ with n being the unknown refractive index of the prism. The difference with the lecture (on the right) is just that we consider refraction for a beam coming from an optically denser medium and entering air. Snell's law applies in the given form, always considering that medium I contains θ_i (incident beam impinging on the considered interface) and medium II contains θ_r of the refracted beam. In the given case: $\theta_r > \theta_i$.

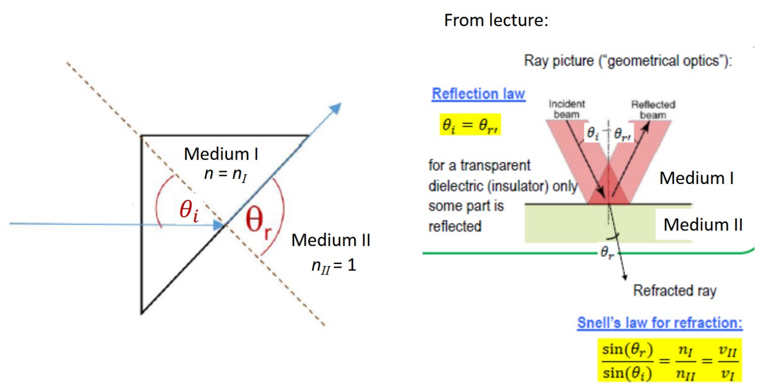


Figure 5: Sketch for the solution 4.

b) Such a value of n is consistent with a quartz prism.

c) We again use the Snell's law. From part (a), $n = \sqrt{2}$. (i) When n is doubled $n_{new} = 2\sqrt{2}$. This means,

$$\frac{\sin\theta_r}{\sin\theta_i} = \frac{n_I}{n_{II}} \Rightarrow \frac{\sin\theta_r}{\sin 45°} = \frac{2\sqrt{2}}{1}$$

This would lead to the value of the $\sin\theta_r$ being greater than 1, which is invalid. Here, the beam gets reflected at the surface, before ever being refracted.

(ii) When n is halved $n_{new} = \frac{1}{\sqrt{2}}$. This means,

$$\frac{\sin\theta_r}{\sin\theta_i} = \frac{n_I}{n_{II}} \Rightarrow \frac{\sin\theta_r}{\sin 45°} = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta_r = \frac{1}{2} \Rightarrow \theta_r = 30°$$

Solution 3.

a) Consider a Gaussian surface in the form of a cylinder with radius r and length l, coaxial with the cylinders. Inside the inner cylinder ($r < a$) and outside the outer cylinder ($r > b$) no charge is enclosed and hence the field is 0. In between the two cylinders ($a < r < b$) the charge enclosed by the Gaussian surface is $-Q$, the total flux through the Gaussian cylinder is

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = E(2\pi rl) \tag{1}$$

Thus, Gauss's law leads to $E(2\pi rl) = \frac{q_{enc}}{\epsilon_0}$, or

$$\vec{E} = \frac{q_{\text{enc}}}{2\pi\epsilon_0 rl}\hat{r} = -\frac{Q}{2\pi\epsilon_0 rl}\hat{r} \text{ (inward) for } a < r < b, \text{ 0 elsewhere} \tag{2}$$

b) Just as with the E field, the enclosed current I_{enc} in the Ampere's loop with radius r is zero inside the inner cylinder ($r < a$) and outside the outer cylinder ($r > b$) and hence the field there is 0. In between the two cylinders ($a < r < b$) the current enclosed is $-I$.

Applying Ampere's law,

$$\int_\Gamma \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 I_{\text{enc}},$$

we obtain

$$\vec{B} = -\frac{\mu_0 I}{2\pi r}\hat{\phi} \text{ (clockwise viewing from the left side) for } a < r < b, \text{ 0 elsewhere} \tag{3}$$

c) For $a < r < b$, the Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0}(-\frac{Q}{2\pi\epsilon_0 rl}\hat{r}) \times (-\frac{\mu_0 I}{2\pi r}\hat{\phi}) = \frac{QI}{4\pi^2\epsilon_0 r^2 l}\hat{k} \text{ (from right to left)} \tag{4}$$

On the other hand, for $r < a$ and $r > b$, we have $\vec{S} = 0$.

d) With $d\vec{A} = (2\pi r dr)\hat{k}$, the power is

$$P = \oiint_S \vec{S}.d\vec{A} = \frac{QI}{4\pi^2\epsilon_0 l}\int_b^a \frac{1}{r^2}(2\pi r dr) = \frac{QI}{2\pi\epsilon_0 l} ln(\frac{b}{a}) \tag{5}$$

e) Since

$$\varepsilon = \int \vec{E}.d\vec{s} = \int_b^a \frac{Q}{2\pi rl\epsilon_0}dr = \frac{Q}{2\pi l\epsilon_0} ln(\frac{b}{a}) = IR \tag{6}$$

the charge Q is related to the resistance R by $Q = \frac{2\pi\epsilon_0 lIR}{ln(b/a)}$. The above expression for P becomes

$$P = \frac{2\pi\epsilon_0 lIR}{ln(b/a)}\frac{I}{2\pi\epsilon_0 l} ln(\frac{b}{a}) = I^2 R \tag{7}$$

which is equal to the rate of energy dissipation in a resistor with resistance R.

Solution 4.

a) We calculate the magnetic field $\vec{B}_1$ due to I_1 and then analyze the forces acting on positions 1 and 2. We refer to Fig. 6: $\vec{B}_1 = \vec{e}_\theta \frac{\mu_0 I_1}{2\pi r}$, with $r^2 = h^2 + (a/2)^2$. This is the known result of a very long wire.

We consider an infinitesimal fraction $d\vec{l}$ of the square loop: at position 2 the force acting on it is $d\vec{F}_2 = I_2 d\vec{l} \times \vec{B}_1$. We integrate over the length a of the relevant segment to find the total force at position 2:

$\vec{F}_2 = I_2 a \frac{\mu_0 I_1}{2\pi r}(-\vec{e}_{r_2})$.

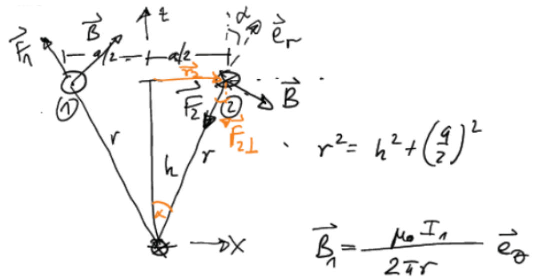


Figure 6: Sketch of the problem with relevant parameters to conduct the vectorial analysis of forces and torques.

Analogously we find the force acting at position 1:

$\vec{F}_1 = I_2 a \frac{\mu_0 I_1}{2\pi r}(+\vec{e}_{r_1})$.

We define the radial unit vectors as follows: $\vec{e}_{r_1} = (-\frac{a/2}{r}, 0, \frac{h}{r})$ and $\vec{e}_{r_2} = (\frac{a/2}{r}, 0, \frac{h}{r})$.

Using the forces $\vec{F}_1$ and $\vec{F}_2$ we find the torques:

$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = r_2 F_{2\perp} \vec{e}_y$ (pointing into the paper), with $F_{2\perp} = F_2[\cos(\alpha)] = F_2[h/r]$.

The torque is collinear with the y-axis: the force lies in the x, z-plane and $\vec{r}_2$ is directed along x.

We obtain that:
$\vec{\tau}_2 = \frac{a}{2} F_2 \frac{h}{r} \vec{e}_y = \frac{a}{2} \frac{a\mu_0 I_1 I_2}{2\pi\sqrt{h^2+(a/2)^2}} \frac{h}{\sqrt{h^2+(a/2)^2}} \vec{e}_y = \frac{\mu_0 I_1 I_2 a^2 h}{4\pi[h^2+(a/2)^2]} \vec{e}_y$

We note that $\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = \vec{\tau}_2$.

The total torque is the vector sum: $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = 2\vec{\tau}_2$

b) The total force on the string is $\vec{F} = \vec{F}_1 + \vec{F}_2 = 2F_{2,||}(-\vec{e}_x)$ because the z-components cancel out. We need to compute $F_{2,||}$.

$F_{2,||} = \sin(\alpha)|\vec{F}_2| = \frac{a/2}{\sqrt{h^2+(a/2)^2}} |\vec{F}_2|$.

We conclude that $\vec{F} = \frac{2\mu_0 I_1 I_2 a^2}{4\pi[h^2+(a/2)^2]}(-\vec{e}_x)$

c) The square loop rotates in the clockwise direction as the total torque is directed along $+y$-direction.

Solution 5.

We use Gauss's law to compute to separately the electric field generated by the large sphere $\vec{E}_\text{l}$ and the one generated by the small sphere $\vec{E}_\text{s}$.

For the large sphere we need to consider $x > R_\text{l}$. The Gauss's law reads $\oiint \vec{E}_\text{l} \cdot d\vec{a} = Q/\epsilon_0$. Q represents in this case the total charge that is uniformly distributed in the large sphere. The vectors $\vec{E}_\text{l}$ and $d\vec{a}$ are parallel to each other and point towards the same direction. The electric field has the same magnitude everywhere on the Gaussian surface and can be taken out of the integral. Therefore Gauss's law simplifies to:
$E_\text{l} \oiint da = E_\text{l} 4\pi x^2 = Q/\epsilon_0$.

We now express Q in terms of the charge density: $Q = \rho_\text{C} \frac{4}{3}\pi R_\text{l}^3$.

The electric field of the large sphere at a point x_p along the x-axis with $x_p = x > R_\text{l}$ is then:
$\vec{E}_\text{l}(x) = \hat{x} \frac{1}{4\pi x^2} Q/\epsilon_0 = \hat{x} \frac{\rho_\text{C} \frac{4}{3}\pi R_\text{l}^3}{4\pi\epsilon_0 x^2}$.

Using the same approach we compute the electric field due to the small sphere at the point $x_p = x$. Here we need to consider that this point lies inside the sphere volume. Making use of Fig. 7 we observe that $x_s < R_s$. Gauss's law is $\oiint \vec{E}_\text{s} \cdot d\vec{a} = q_\text{enc}/\epsilon_0$. The enclosed charge reads $q_\text{enc} = -\rho_\text{C} \frac{4}{3}\pi x_s^3$.

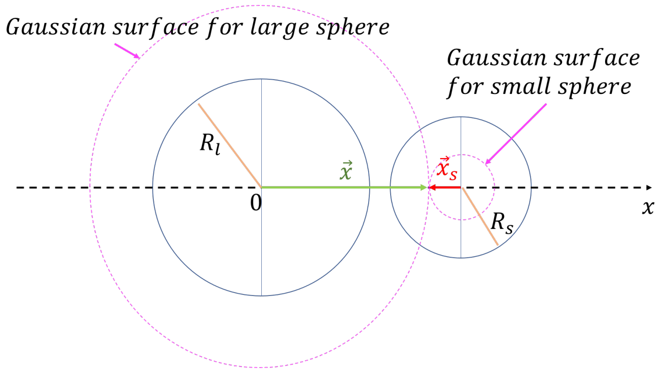


Figure 7: Sketch of the problem geometry.

$\vec{E}_\mathrm{s}(x) = \hat{x}\frac{1}{4\pi x_s^2} q_\mathrm{enc}/\epsilon_0 = \hat{x}\frac{-\rho_\mathrm{C}\frac{4}{3}\pi x_s^3}{4\pi\epsilon_0 x_s^2}$. Now we need to express x_s in a way consistent with the coordinate system of the large sphere: we notice that $|x_s| = |x-d|$.

The formula for $\vec{E}_s(x)$ is rewritten into: $\vec{E}_\mathrm{s}(x) = \hat{x}\frac{-\rho_\mathrm{C}\frac{4}{3}\pi(x-d)^3}{4\pi\epsilon_0|x-d|^2} = \hat{x}\frac{-\rho_\mathrm{C}(x-d)^3}{3\epsilon_0|x-d|^2}$.

By superposition principle the total electric field is:
$\vec{E}_\mathrm{tot}(x) = \vec{E}_\mathrm{l}(x) + \vec{E}_\mathrm{s}(x) = \left(\frac{\rho_\mathrm{C} R_\mathrm{l}^3}{3\epsilon_0 x^2} - \frac{\rho_\mathrm{C}(x-d)^3}{3\epsilon_0|x-d|^2}\right)\hat{x}$

Solution 6.

a) At the critical angle $\theta_{3,\mathrm{crit}}$ the angle of refraction is 90°. Applying Snell's law gives $\sin\theta_{3,\mathrm{crit}} = n_\mathrm{B}/n_\mathrm{A} \Rightarrow \theta_{3,\mathrm{crit}} = \sin^{-1}\left(\frac{n_\mathrm{B}}{n_\mathrm{A}}\right) = \sin^{-1}\left(\frac{1.440}{1.480}\right)^{-1} = 76.7°$.

b) There is total internal reflection for:

$$90° \geq \theta_3 \geq \theta_{3,\mathrm{crit}} \qquad \theta_2 = 90° - \theta_3 \tag{8}$$

Additionally, we have $\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_\mathrm{A}}{1} \Rightarrow \theta_{1,\mathrm{crit}} = \sin^{-1}(n_\mathrm{A}\sin(90° - \theta_3)) = 20.0°$. The relevant range of θ_1 for total internal reflection is therefore: $0 \leq \theta_1 \leq 20.0°$.

c) Due to the law of reflection $\theta_\mathrm{r} = \theta_\mathrm{i}$, the incident angle θ_3 for total internal reflection at the top material A/material B interface is also the relevant incident angle for the bottom interface: yes, the light will be totally internally reflected at both interfaces.