Exercise sheet 13: Propagating and standing waves, Poynting vector, superposition

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11/12/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Radio station/Category II (After training for solution: 25 min))

A radio station (rs) is allowed to broadcast at a maximum average power of 25 kW radially. If an electric field amplitude of 0.020 V/m is considered to be acceptable for receiving the radio transmission with a relevant signal strength, estimate how many kilometers away you might be able to hear this station in your radio. Assume a point-like source which emits a spherical wave. Integrate $\vec{S}$ over an appropriately chosen surface.

Exercise 2.

(Poynting vector in a capacitor/Category II (After training for solution: 20 min))

a) Show that the so-called Poynting vector $\vec{S} = \frac{1}{\mu_0}(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t))$, which describes the energy flux density, points radially inwards toward the center of a circular parallel–plate capacitor when it is being charged [this means: $E = E(t)$].

b) Integrate $\vec{S}$ over the cylindrical boundary of the capacitor gap to obtain the energy flux and show that the rate at which energy enters the capacitor via the Poynting vector $\vec{S}$ is equal to the rate at which electrostatic energy U is being stored in the electric field of the capacitor. Ignore fringing fields of $\vec{E}$ to show $-\oiint \vec{S} \cdot d\vec{a} = \frac{dU}{dt}$.

Exercise 3.

(Energy flow for a standing wave/Category II (After training for solution: 35 min))

Consider the standing electromagnetic wave from the lecture given by $\vec{E} = 2\hat{z}E_0[\sin(ky)\cos(\omega t)]$ and $\vec{B} = -2\hat{x}\frac{E_0}{c}[\cos(ky)\sin(\omega t)]$.

a) Calculate the time-dependent energy densities $u_E(y,t)$ and $u_B(y,t)$. Draw plots of the densities at ωt values of 0, $\pi/4$, $\pi/2$, and $3\pi/4$.

b) Calculate the y component of the time-dependent Poynting vector, $S_y(y,t)$, and plot its value at different times corresponding to ωt equal to 0, $\pi/4$, $\pi/2$ and $3\pi/4$. Are these plots consistent with how the energy densities vary as a function of time t?

c) How large is the time-averaged Poynting vector?

Exercise 4.

(Confined waves/Category II (After training for solution: 15 min))

Consider a string with linear mass density σ, length L and tension T. We neglect gravitational force on the string. The string is along the $\hat{x}$ direction, starting at $x = 0$, ending at $x = L$. We consider small deformations $\xi(x,t)$ along the string. The ends of the string are rigidly fixed to an extremely heavy wall. Consider the wave equation obeyed by $\xi(x,t)$ as discussed in the lecture. The general solution to this equation is the superposition of left- and right-propagating waves. We assume that all of those waves have the same amplitude ξ_0 and phase velocity v.

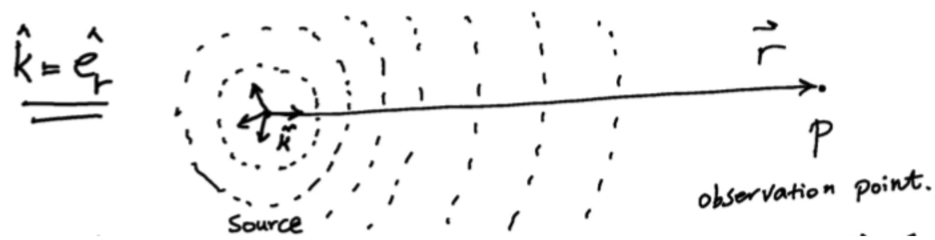
a) Explain why the boundary conditions for $\xi(x,t)$ impose that the wavelength of left- and right-propagating waves must be of the form $\lambda_i = \frac{2L}{i}$, $i = 1, 2, ...$, in order to lead to constructive interference. What are the corresponding wavevectors k_i and frequencies ω_i? The numbers i count the so-called harmonics.

b) Show that the solution $\xi_i(x,t) = 2\xi_0 \sin(k_i x)\cos(\omega_i t)$ fulfills the wave equation and boundary conditions.

c) Consider a guitar with a steel string fixed rigidly between two points. When excited, it has a number of frequencies ω_i (harmonics) at which it vibrates. Assume that the tension put on the guitar string produces a phase velocity of 470 m/s. The length of the string is 66.5 cm. What are the frequencies $\nu_i = \omega_i/(2\pi)$ of the first ($i = 1$) and second ($i = 2$) harmonic, i.e. the lowest and the second lowest frequencies? Can their sound be heard?

Solution 1.

The magnitude of the Poynting vector is the power per unit area. The energy is assumed to be conserved in the system. Hence, the integral of the Poynting vector over a closed surface around the source must be equal to the power of the source (both averaged over time). In the problem it is assumed that the power is broadcasted radially. Therefore, we assume spherical symmetry for the Poynting vector around the source.



For the solution, we pick a specific point in space $\vec{r}$ for the receiver at a distance r, thereby the propagation direction is given ($\hat{k} = \hat{e}_r$). The electric and magnetic field of the wave can be written as:

$$\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{e}_1,$$
$$\vec{B}(\vec{r},t) = \frac{E_0}{c}\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k}\times\hat{e}_1),$$

where $\hat{e}_1$ is an arbitrary unit vector perpendicular to the propagation direction $\hat{k}$.

The Poynting vector, using $\varepsilon_0\mu_0 = \frac{1}{c^2}$, reads:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{E_0^2}{\mu_0 c}\cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{e}_1\times(\hat{k}\times\hat{e}_1),$$

$$\vec{S} = (\varepsilon_0 c^2)\frac{E_0^2}{c}\cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)[\hat{k}(\hat{e}_1\cdot\hat{e}_1) - (\hat{e}_1\cdot\hat{k})\hat{e}_1],$$

where we used

$$\hat{e}_1\cdot\hat{e}_1 = 1, \hat{e}_1\cdot\hat{k} = 0,$$

$$\vec{S} = \varepsilon_0 c E_0^2\cos^2(k\cdot r - \omega t + \delta)\hat{e}_r.$$

$$P = \oiint_{sphere}\vec{S}\cdot\hat{n}da = \oiint\vec{S}\cdot\hat{e}_r da = \iint(\varepsilon_0 c E_0^2\cos^2(kr - \omega t + \delta)\hat{e}_r)\cdot(\hat{e}_r r^2\sin\theta d\theta d\phi)$$

$$P = 4\pi r^2\varepsilon_0 c E_0^2\cos^2(kr - \omega t + \delta)$$

$$\langle P\rangle = \langle 4\pi r^2\varepsilon_0 c E_0^2\cos^2(kr - \omega t + \delta)\rangle = 4\pi r^2\varepsilon_0 c E_0^2[\frac{1}{T}\int_0^T\cos^2(kr - \frac{2\pi}{T}t + \delta)dt]$$

$$\langle P\rangle = 4\pi r^2(\frac{1}{2}\varepsilon_0 c E_0^2) = 2\pi r^2\varepsilon_0 c E_0^2$$

From here we extract the radius:

$$r = \sqrt{\frac{\langle P\rangle}{2\pi\varepsilon_0 c E_0^2}} = \sqrt{\frac{25\times 10^3\mathrm{W}}{2\pi(8.85\times 10^{-12}\ \mathrm{C}^2/(\mathrm{Nm}^2))(3\times 10^8\ \mathrm{m/s})(0.02\ \mathrm{v/m})^2}} = 61200\ \mathrm{m} = 61.2\ \mathrm{km}.$$

Solution 2.
See Fig. 1 for a schematics on the structure.

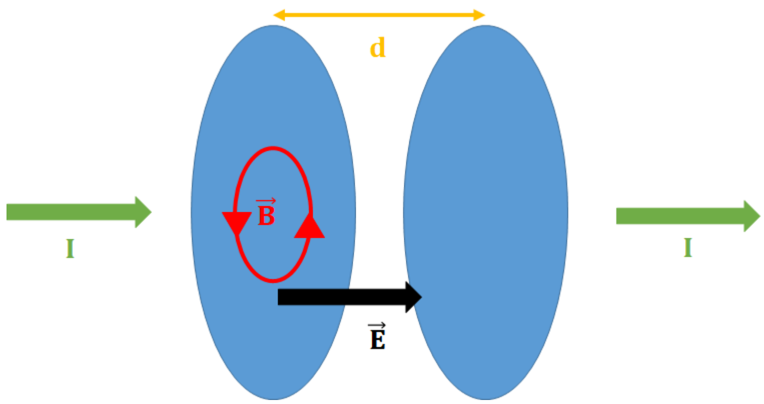


Figure 1: Sketch of a plate capacitor (blue). The radius of one plate is taken to be r_0.

a) For any point between capacitor plates the Poynting vector points toward the axis of the capacitor (Fig. 1): $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$.

b) Using Ampere's law (make reference to solution of exercise 1 from exercise sheet 11 for more details) we find $B2\pi R_{path} = \mu_0\epsilon_0\frac{d(\pi R_{flux}^2 E)}{dt} \to B = \left[\frac{1}{2}\mu_0\epsilon_0\frac{R_{flux}^2}{R_{path}}\frac{dE}{dt}\right]_{R_{flux}=R_{path}=r_0} = \frac{1}{2}\mu_0\epsilon_0 r_0\frac{dE}{dt}$, with r_0 being the radius of the circular plate. Therefore knowing the formula for the Poynting vector and that electric and magnetic fields are orthogonal to each other $S = \mid\vec{S}\mid = \frac{1}{\mu_0}\mid\vec{E}\times\vec{B}\mid = \frac{1}{\mu_0}EB = \frac{1}{\mu_0}E\frac{1}{2}\mu_0\epsilon_0 r_0\frac{dE}{dt} = \frac{1}{2}\epsilon_0 r_0 E\frac{dE}{dt}$.

The energy flow into the capacitor volume V' is given by the flux of $\vec{S}$ calculated via $\oiint\vec{S}\cdot d\vec{a}$ over the closed surface of the considered volume V'. The relevant equation from the lecture is the *continuity equation* in integral form: $-\frac{d}{dt}\iiint_{V'} u_{\mathrm{EM}}dV - \oiint_{\mathrm{surface}(V')}\vec{S}\cdot d\vec{a} = 0$. Here, the first term is an integration performed for the volume in which the relevant energy density u_{EM} is considered (here the inner region of the plate capacitor forming a cylindrical volume V'). The second term evaluates the energy flux across the surface of the considered volume V'. Reordering the equation provides: $-\oiint_{\mathrm{surface}(V')}\vec{S}\cdot d\vec{a} = \frac{d(\iiint_{V'} u_{\mathrm{EM}}dV)}{dt}$. Following the text of the problem we analyze moments in time at which $\vec{S}$ points inwards. The vector quantity $d\vec{a}$ in the surface integral points -by convention- outwards, i.e., the scalar product of $\vec{S}$ and $d\vec{a}$ provides a negative contribution to the integral. The minus-sign in front of the integral makes the inward energy flux overall positive, hence, u_{EM} increases at the relevant moments of time. This mathematical consideration is hence consistent with the text in the problem. The term $\iiint_{V'} u_{\mathrm{EM}}dV$ provides the amount of total energy U being inside the capacitor.

Given a cylindrical volume V' which is consistent with the capacitor volume the Poynting vector is constant on its curved surface and can therefore be taken out of the integral: $-\oiint_{\mathrm{surface}(V')}\vec{S}\cdot d\vec{a} = SA = S2\pi r_0 d = \epsilon_0 d\pi r_0^2 E\frac{dE}{dt}$, where A is the total surface area of the *curved* outer surface of the cylindrical volume (the planar top and bottom surfaces of the cylindrical volume V do not count because $\vec{S}$ and their $d\vec{a}$ are orthogonal and no flux contribution is provided). The amount of stored energy in the capacitor provided by an electric field of strength E reads $U = \left(\frac{1}{2}\epsilon_0 E^2\right)\left(\pi r_0^2 d\right)$. The rate of change of U is $\frac{dU}{dt} = \epsilon_0\pi d r_0^2 E\frac{dE}{dt}$. By comparison of the two quantities one finds $\frac{dU}{dt}|_{V'} = -\oiint_{\mathrm{surface}(V')}\vec{S}\cdot d\vec{a}$ which represents the energy conservation as stated above.

Solution 3.

a) The time dependent energy densities are:

$$u_E = \frac{1}{2}\varepsilon_0 |E|^2 = \frac{1}{2}\varepsilon_0 \left(2E_0\right)^2 \sin^2\left(\frac{2\pi y}{\lambda}\right)\cos^2\left(\frac{2\pi ct}{\lambda}\right) = u_{E,m}\sin^2\left(\frac{2\pi y}{\lambda}\right)\cos^2\left(\frac{2\pi ct}{\lambda}\right)$$

$$u_B = \frac{1}{2\mu_0}|B|^2 = \frac{1}{2\mu_0}\left(\frac{2E_0}{c}\right)^2 \cos^2\left(\frac{2\pi y}{\lambda}\right)\sin^2\left(\frac{2\pi ct}{\lambda}\right) = u_{B,m}\cos^2\left(\frac{2\pi y}{\lambda}\right)\sin^2\left(\frac{2\pi ct}{\lambda}\right)$$

where, $u_{E,m}$ and $u_{B,m}$ are maximum electric and magnetic energy densities which are ploted in Fig. 2 (a) and (b), respectively.

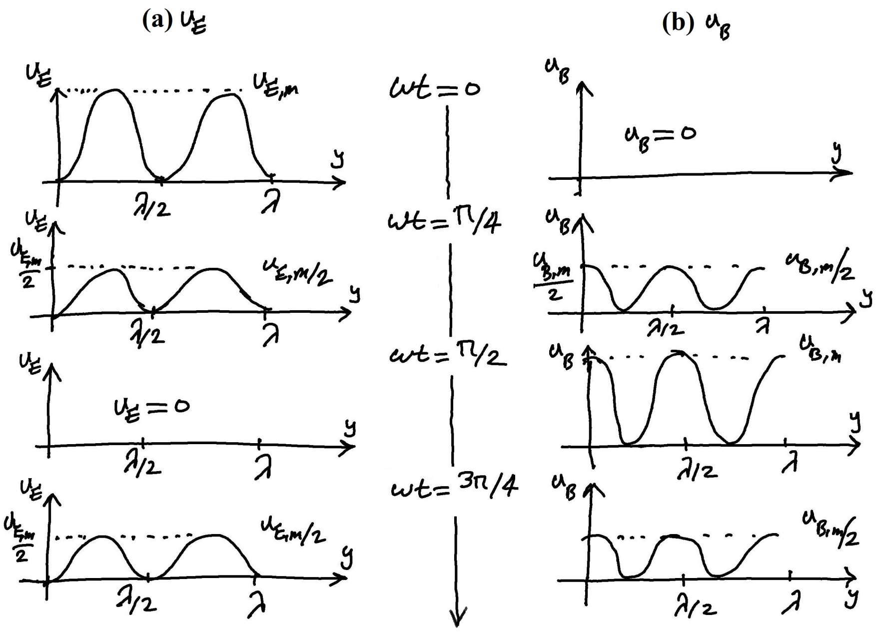


Figure 2: (a) electric and (b) magnetic energy densities for the standing wave.

b) For the Poynting vector $\vec{S}$ we have:

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left[\left\{\hat{z}\,2E_0\sin\left(\frac{2\pi y}{\lambda}\right)\cos\left(\frac{2\pi ct}{\lambda}\right)\right\}\times\left\{(-\hat{x})\frac{2E_0}{c}\cos\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi ct}{\lambda}\right)\right\}\right]$$

$$\vec{S} = \frac{4E_0^2}{\mu_0 c}\sin\left(\frac{2\pi y}{\lambda}\right)\cos\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi ct}{\lambda}\right)\cos\left(\frac{2\pi ct}{\lambda}\right)(-\hat{y})$$

$$\vec{S} = (-\hat{y})\,\frac{E_0^2}{\mu_0 c}\,\underbrace{\sin\left(2\frac{2\pi y}{\lambda}\right)}_{\text{term ①}}\underbrace{\sin\left(2\frac{2\pi ct}{\lambda}\right)}_{\text{term ②}}$$

Term ① oscillates between -1 and +1 as a function of y (at fixed time t), but at half of the wavelength of the electromagnetic wave as it is shown in Fig. 3.

Term ② oscillates between -1 and +1 as a function of time, t, (at fixed y), but at twice the frequency of the electromagnetic wave as it is shown in Fig. 4.

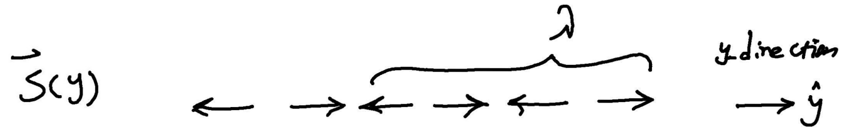


Figure 3: Spatial dependence of the Poynting vector of the standing wave.

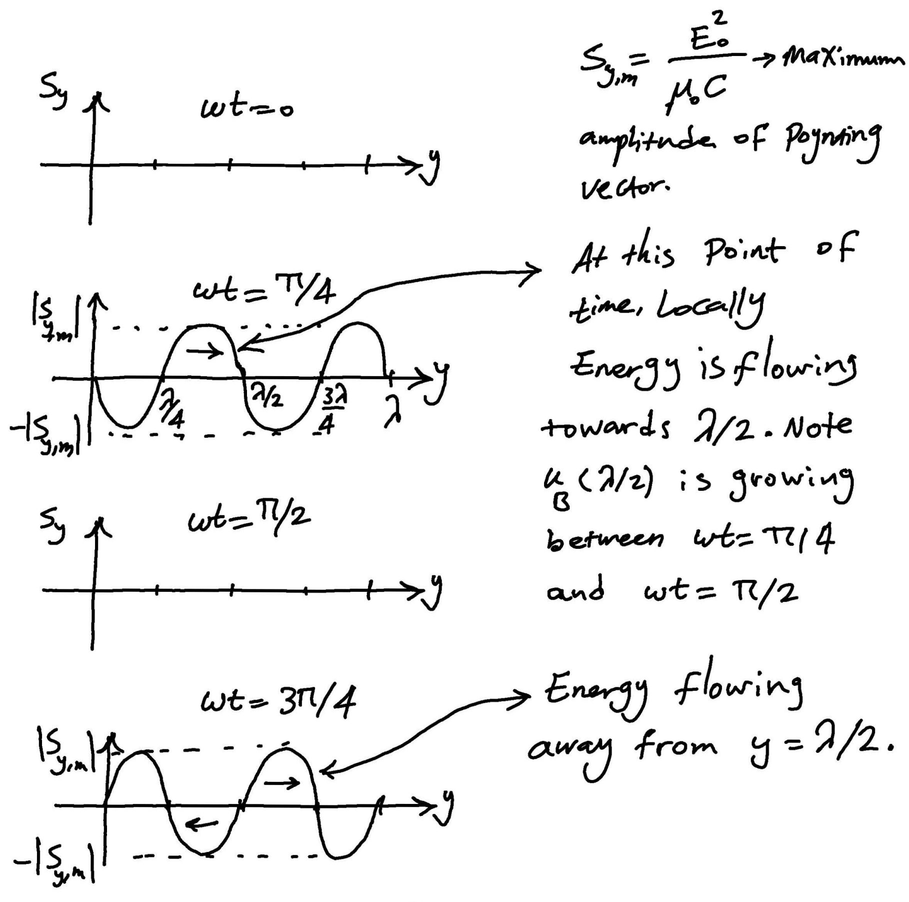


Figure 4: Time dependence of the Poynting vector of the standing wave.

c) Time-averaged Poynting vector, $\left\langle|\vec{S}|\right\rangle$, is:

$$\left\langle|\vec{S}|\right\rangle = \frac{E_0^2}{\mu_0 c}\ \sin\left(2\frac{2\pi y}{\lambda}\right)\left\langle\sin\left(2\frac{2\pi ct}{\lambda}\right)\right\rangle = 0$$

The time energy transferred into a specific direction is zero and is stored in place.

Solution 4.

a) The ends of the string are rigidly fixed to an extremely heavy wall. Therefore, the wave has nodes at $x = 0$ and $x = L$ which leads to (i) $\xi(0,t) == 0$ and (ii) $\xi(L,t) = 0$ for any time t.

To have constructive interference the length L of the string must be a multiple of half of the wavelength $\rightarrow L = i\frac{\lambda}{2}$ for $i = 1, 2, 3, 4....$ This leads to the following condition for λ:

$\lambda_i = \frac{2L}{i}$ (Fig. 5) From this condition it follows:

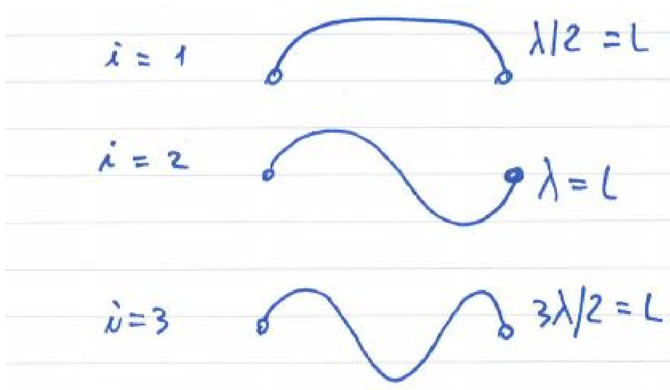


Figure 5: Solution 4a - sketch for first three modes of confined waves in the string

$k_i = \frac{2\pi}{\lambda_i} = 2\pi\frac{i}{2L} = i\frac{\pi}{L}$ and

$\omega_i = vk_i = v\frac{i\pi}{L}$

From the lecture notes we know that $v = \sqrt{\left(\frac{T}{\sigma}\right)}$

b) The wave equation is $\frac{\partial^2\xi_i(x,t)}{\partial t^2} = v^2\frac{\partial^2\xi_i(x,t)}{\partial x^2}$. We plug into this equation the expression for $\xi_i = \xi_i(x,t)$. The wave equation becomes:

$\frac{\partial^2}{\partial t^2}(2\xi_0\sin(k_ix)\cos(\omega_it)) = v^2\frac{\partial^2}{\partial x^2}(2\xi_0\sin(k_ix)\cos(\omega_it))$

$((-\omega_i^2)2\xi_0\sin(k_ix)\cos(\omega_it)) = v^2((-k_i^2)2\xi_0\sin(k_ix)\cos(\omega_it))$

It follows that:
$-\omega_i^2\xi_i(x,t) = -v^2k_i^2\xi_i(x,t)$ hence

$\omega_i^2 = v^2k_i^2 \rightarrow \omega_i = vk_i$

The boundary conditions then becomes (i) $\xi(0,t) = 2\xi_0\sin(k_i0)\cos(\omega_it) = 0$ this is verified because of $\sin(k_i0) = 0$ and (ii) $\xi(0,t) = 2\xi_0\sin(k_iL)\cos(\omega_it) = 0 \rightarrow \sin(k_iL) = 0$ this is verified because of $k_iL = i\pi$.

c) $v = 470$ m/s; $L = 0.665$ m; $k_1 = \frac{\pi}{L}$ for i=1 and $k_2 = \frac{2\pi}{L}$ for i=2

$\lambda\nu = v$. Therefore it follows that $\nu_1 = \frac{v}{\lambda_1}$= 353.4 Hz and $\nu_2 = \frac{v}{\lambda_2} = 706.8$ Hz. These sounds can be heard as the frequency is above 20 Hz and below 20 kHz.