Exercise sheet 12: Transformer, waves

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4/12/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Transformer/Category I)
A transformer (battery charger) is used to charge ten nickel-cadmium batteries connected in series (total emf of 12.5 V DC). The transformer needs to have a voltage difference $\Delta V_2 = V_{\text{rms},2} = 15.0$ V at its output (secondary coil) to charge these batteries. It uses a step-down transformer with a 200-loop primary coil and $\Delta V_1 = V_{\text{rms},1} = 120$ V in the input.

a) How many loops should there be in the secondary coil?

b) If the charging current is $I_{\text{rms},2} = 16.0$ A, what is the input current $I_{\text{rms},1}$ in the primary coil?

Exercise 2.
(Electromagnetic wave/Category I)
The electric field of an electromagnetic wave is given by $E_x = E_0 \cos(kz + \omega t)$, $E_y = E_z = 0$ in Cartesian coordinates. Determine (a) the direction of propagation and (b) the direction of $\vec{B}$. Hint: for an electromagnetic wave the vectors $\vec{E}$, $\vec{B}$ and propagation vector $\vec{k}$ *must* all be orthogonal to each other, following the right hand rule: $\vec{E}$ - thumb, $\vec{B}$ - pointer finger, $\vec{k}$ - middle finger.

Exercise 3.
(String under tension/Category I)
The wave function for a travelling transverse wave on a string under tension is given in SI units (t in seconds, x in meters, ω in rad/s) as follows: $\vec{\xi}(x,t) = 0.450 \sin(10\pi t - 3\pi x + \frac{\pi}{4})\hat{y}$.

a) What is the speed v of the wave collinear with the x-axis?

b) In which direction does the wave travel?

c) What is the wavelength?

d) What is the frequency?

e) What is the vertical position of a mass element dm of the string at $t = 0$ s and $x = 0.100$ m?

f) What is the maximum transverse speed v_y of an element of the string in y-direction?

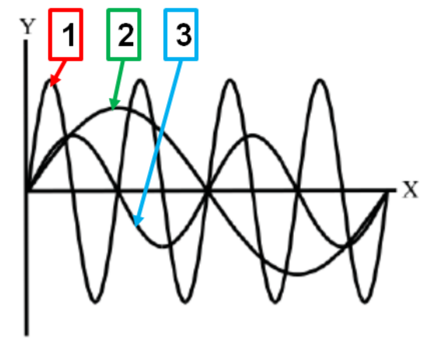
Exercise 4.
(Waves on different strings/Catgeory I)

The figure shows a snapshot (a photo taken at a given time) of three waves propagating along some strings. The waves come from $-\infty$ and go to $+\infty$. Associate each wave with one of the following expressions:

a) $y(x,t) = y_a \sin(2x - 4t)$

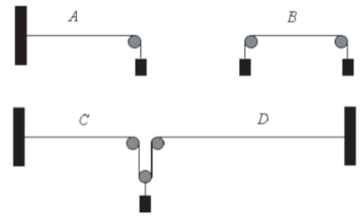
b) $y(x,t) = y_b \sin(4x - 8t)$

c) $y(x,t) = y_c \sin(8x - 16t)$



What property do those waves have in common?

Exercise 5.
(Waves on four strings/Catgeory I)



The figure shows four strings A to D. Tension T is applied to them using one or two weights of mass m. Each weight has the same mass m. Strings A,B, and C have the same linear mass density σ_s; string D has a larger mass density σ_l. Order the strings according to the phase velocity v of the waves that propagate along them. Start with the largest velocity.

Solution 1.

a) The solution is given by the relationship of input (ΔV_1, primary circuit) and output voltage differences (ΔV_2, secondary circuit) for a transformer: $\frac{\Delta V_2}{\Delta V_1} = \frac{N_2}{N_1}$, where N_i with $i = 1, 2$ are the numbers of turns of coils i. Therefore the number of turns reads: $N_2 = \frac{\Delta V_2}{\Delta V_1} N_1 = \frac{15\,\text{V}}{120\,\text{V}} 200 = 25$. These relations work with both quantities consistently considered on both sides: the amplitude of voltages $\Delta V = V_{0i}$ and rms-values.

b) The current in the primary circuit (input current) is obtained by considering the energy conservation in the transformer $\Delta V_1 I_1 = \Delta V_2 I_2$ and $\frac{\Delta V_2}{\Delta V_1} = \frac{N_2}{N_1}$. Again these relations work with both quantities consistently considered on both sides: the amplitudes of voltages $\Delta V = V_{0i}$ and currents $I_{0,i}$ or their rms-values. Combination of the two equations leads to $\frac{I_2}{I_1} = \frac{N_1}{N_2} \Rightarrow I_1 = \frac{N_2}{N_1} I_2 = \frac{25}{200} 16$ A $= 2$ A.

Solution 2.

a) The electric field propagates along $\hat{k} = -\hat{z}$, with $\hat{k}$ being the unit vector giving the direction of the wavevector.

b) The magnetic field direction is given by $\hat{k} \times \hat{E} = (-\hat{z}) \times \hat{x} = -\hat{y}$.

Solution 3.

a) As indicated in the question, we assume SI units throughout this question. To start, note that the given wave can be rewritten as: $\vec{\xi}(x,t) = 0.450 \sin(10\pi t - 3\pi x + \frac{\pi}{4})\hat{y} = 0.450 \sin\left(-(3\pi x - 10\pi t - \frac{\pi}{4})\right)\hat{y}$
$= 0.450 \sin\left(3\pi x - 10\pi t + \frac{3\pi}{4}\right)\hat{y}$, where we used $\sin(-x) = \sin(x + \pi)$.

A solution describing a wave is $\vec{\xi}(x,t) = \xi_0 \sin(\vec{k} \cdot \vec{x} - \omega t + \delta)\hat{n} = \xi_0 \sin[k(x - vt) + \delta]\hat{n}$ where in the last step $\vec{k} \parallel \vec{x}$ is assumed. Comparing with $\vec{\xi}(x,t) = 0.450 \sin\left(3\pi x - 10\pi t + \frac{3\pi}{4}\right)\hat{y}$, we identify $k = 3\pi$ rad/m and $\omega = kv = 10\pi$ rad/s so $\Rightarrow v = \frac{\omega}{k} = \frac{10\pi}{3\pi} \frac{\text{rad/s}}{\text{rad/m}} = \frac{10}{3}$ m/s.

b) We are interested in which direction a given displacement of the wave phase propagates. To find this, we set the phase to a constant and derive an expression for its position as a function of time:
$kx - \omega t + \delta = C_{const}$
$kx = \omega t - \delta + C_{const}$
$\Rightarrow x = x(t) = \frac{\omega}{k} t - \frac{1}{k}(\delta + C_{const})$
The last term on the right hand side is constant and does not influence the direction of propagation. Substituting the values for ω, k we find $x(t) = \frac{10}{3} \frac{\text{m}}{\text{s}} t - \frac{1}{k}(\delta + C_{const})$. The wave travels towards positive x.

c) General expression for a wave $\xi(x,t) = \xi_0 \sin(kx - \omega t + \delta)$.We are concerned only with the absolute value of k. $\Rightarrow k = 3\pi$ rad/m with $k = 2\pi/\lambda \Rightarrow 3\pi = 2\pi/\lambda \Rightarrow \lambda = \frac{2}{3}$m.

d) $\omega = 10\pi$ 1/s$, \omega = 2\pi f \Rightarrow 2\pi f = 10\pi$ 1/s $\Rightarrow f = \frac{10}{2}\frac{1}{s} = 5$ Hz.

e) $\vec{\xi}(x = 0.1\text{ m}, t = 0\text{ s}) = 0.45\hat{y} \sin(0 - 3\pi \cdot 0.1 + \pi/4)$ m
$\vec{\xi}(x = 0.1\text{ m}, t = 0\text{ s}) = \hat{y}0.45 \sin(\pi(-0.3 + 1/4))$ m $= \hat{y}0.45(-0.156)$ m $= -\hat{y}$ 0.0704 m. $|\vec{\xi}| = 0.0704$ m.

f) $\vec{v_y} = \frac{\partial \xi(x,t)}{\partial t} = \hat{y}0.45 \frac{\partial}{\partial t}[\sin(10\pi t - 3\pi x + \pi/4)]$
$\vec{v_y} = \hat{y}0.45 \cdot 10\pi \cdot \cos(10\pi t - 3\pi x + \pi/4)$ m/s
$\vec{v_y} = \hat{y}4.5\pi \cdot \cos(10\pi t - 3\pi x + \pi/4)$ m/s
$|\vec{v_y}|_{max} = 4.5\pi$ m/s $= 14.14$ m/s.

Solution 4.

The letters will be matched to the numbers labeling the waves as in Fig. 1. We suggest to consider x in units of m and t in units of s. The correct matching is the following: (a) with 2, (b) with 3 and (c) with 1. Explanation: The general

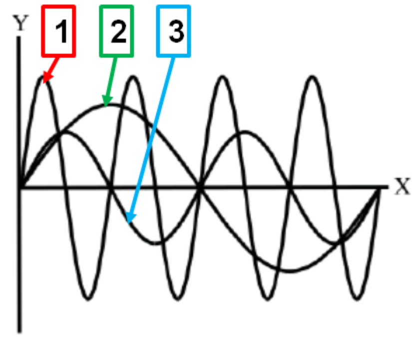


Figure 1: Sketch of different waves with numeric labels

solution for a right moving wave with index i described by a sine-function is given by: $y(x,t) = y_i \sin(k_i x - \omega_i t + \delta_{0,i})$ where $\delta_{0,i}$ is some starting phase. Assuming t_0 to be the time at which the image was taken we conclude that for $t = t_0$ the following relation can be assumed $-\omega_i t_0 + \delta_{0,i} = 0$ for all displayed cases as all curves start with zero amplitude at $x = 0$ (or $k_i x = 0$).

The spatial dependence of the displayed curves is thus described by $y(x,t) = y_i \sin(k_i x)$ at the time $t = t_0$. We also know that $k_i x = \frac{2\pi}{\lambda_i} x$ with λ_i being the wavelength (spatial period). For the different waves i we find:

(a) $2 = \frac{2\pi}{\lambda_a} \Rightarrow \lambda_a = \pi$ (for the frequency one reads: $\omega_a = 4$, needed for the further question)

(b) $4 = \frac{2\pi}{\lambda_b} \Rightarrow \lambda_b = \pi/2$ (for the frequency one reads: $\omega_b = 8$)

(c) $8 = \frac{2\pi}{\lambda_c} \Rightarrow \lambda_c = \pi/4$ (for the frequency one reads: $\omega_c = 16$)

Therefore (a) must correspond to the illustrated wave with the longest period and (c) must correspond to the wave with the shortest one. The period of wave (b) is between the other two values.

Analyzing the phase velocities $v_i = \lambda_i \cdot \omega_i/(2\pi)$ of the waves one finds that all of them have the same velocity $v_i = 2$ (following our assumption).

Solution 5.

The velocity is given by $v = \sqrt{\frac{T}{\sigma}}$, where T is the tension and σ is the linear mass density.

Weights and hence tensions are the same in cases A and B, i.e., $T_A = T_B$, and $\sigma_{s,A} = \sigma_{s,B}$; thus we get $v_A = v_B$. Because of the different arrangement of weight and strings labelled C and D, we analyze for the corresponding tensions: $T_C = T_D = \frac{1}{2} T_A$. Since $\sigma_{s,A} = \sigma_{s,C}$ we find $v_C < v_A$. String D has the largest linear mass density σ_l and the smallest tension T_D. A wave on string D has the minimum velocity of all scenarios.

To summarize: $v_A = v_B > v_C > v_D$.