Exercise sheet 11: Displacement current, magnetic energy

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27/11/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Magnetic field from displacement current/Category I)

Suppose that a circular parallel–plate capacitor has a radius of $R_0 = 3.0$ cm and a plate separation of $d = 5.0$ mm. A sinusoidal potential difference $V = V_0 \sin(2\pi ft)$ is applied across the plates, where $V_0 = 150$ V and $f = 60$ Hz. Ignore fringing fields of E. Hint: This exercise is meant to show that fields from the term dE/dt in a capacitor are small at 60 Hz, i.e., when AC currents/electric fields vary slowly.

a) In the region between the plates, show that the magnitude of the induced magnetic field is given by $B = B_0(R)\cos(2\pi ft)$, where R is the radial distance from the capacitor's central axis.

b) Determine the expression for the amplitude $B_0(R)$ of this time-dependent (sinusoidal) field when $R \leq R_0$ and when $R > R_0$.

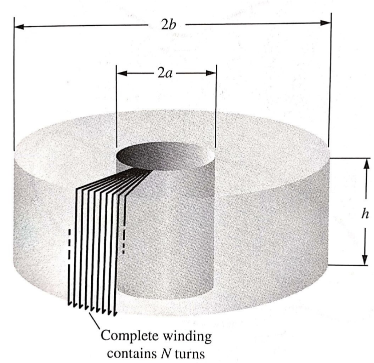
c) Plot $B_0(R)$ in tesla for the range $0 \leq R \leq 10$ cm.

d) Now assume that a wire connected to the capacitor carries exactly the amount of absolute charge current I as given by the maximum displacement current I_D flowing through the whole cross-section of the capacitor in its gap. The wire is assumed to be infinitely long and has a diameter of 2 mm. Calculate the magnetic field generated by I just at its surface in the limit of an infinitely long wire. Compare this value with the field B in the gap of the capacitor at $R = 1$ mm. How much weaker is this field generated by the displacement current in the capacitor compared to the field of the charge-based current?

Exercise 2.

[Self-inductance and magnetic energy in toroidal solenoid /Category II (After training for solution: 25 min)]

Consider a toroidal solenoid (coil) with the geometrical parameters presented in the figure. Complete winding contains N turns all around the core.



Toroidal coil of rectangular cross section. Only a few turns are shown.

a) Calculate the self-inductance, L, of the system.

b) Obtain the magnetic energy density, u, stored in the system.

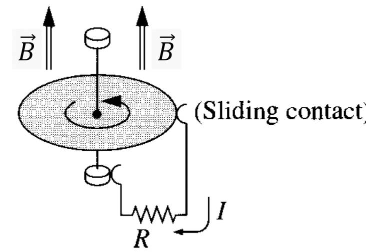
c) Calculate the total energy, U, stored in the system.

d) Evaluate L for a toroidal solenoid with dimensions of $2a = 10$ mm, $2b = 15$ mm and $h = 5$ mm with $N = 100$ turns of wire. How much magnetic energy is stored in the system if one applies a current of $I = 3$ A to the solenoid. Suppose that we store the same amount of energy on a circular plate capacitor with plate diameter of $R = 2b = 15$ mm and distance between plates $d = h = 5$ mm filled with a dielectric with dielectric constant of 5. How much voltage, V, needs to be applied?

Exercise 3.

(Faraday's disk / Category II (After training time needed: 20-25 min))

Faraday's disc was the first electromagnetic generator. The working principle is as follows: A conductor disc (metal) of radius a rotates with a constant angular velocity ω about a vertical axis, through a uniform field $\vec{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the edge of the disk (see sketch).



a) Find the expression for the current in the resistor. Hint: When the disk rotates the charges in the conductor inherit a velocity vector.

b) Now let's consider to construct a bicycle dynamo using Faraday's disk. Instead of the resistor, we power a lightbulb with an electrical power consumption $P_{\text{el.}} = 4$ W and resistance $R = 1\ \Omega$. Due to size restrictions, we choose a disk radius of $a = 2$ cm. Using permanent magnets, we provide $B = 0.2$ T. Calculate the frequency of rotation f (turns per second, $f = \omega/2\pi$) needed to supply 4 W to the bulb. Is the required frequency realistic for the application as a bicycle dynamo?

Exercise 4.

(Bicycle dynamo / Category II (After training time needed: 15-20 min))

Consider a commercial bicycle dynamo as sketched below. Instead of spinning a coil in a magnetic field (compare with Exercise 3), a permanent magnet is spinning with respect to a fixed coil (solenoid). The part labeled "Soft iron core" allows one to guide the magnetic field lines of the rotating magnet through the "fixed coil". As a consequence one can assume a magnetic field inside the coil which amounts to $B(t) = B_0 \sin(\omega t)$ with $B_0 = 0.2$ T. The coil has $N = 1000$ turns and all the turns have a radius $a = 2$ cm.

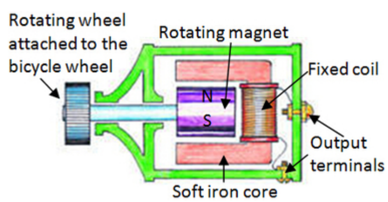


Figure 1: Bicycle Dynamo.

a) Calculate the frequency of rotation f (turns per second, $f = \omega/2\pi$) needed to apply a time-averaged electric power $P_{\text{avg}} = 4$ W to the bulb with $R = 1\ \Omega$. (Here we neglect any self-inductance of the coil. Time-averaged electric power means that one averages the time-dependent dissipated power $P(t)$ over one period T. The time-averaged power P_{avg} is found to be $emf_0^2/2R$ where emf_0 is the amplitude of the time-dependent electromotive force.)

b) Plausibility check (optional): A cyclist drives with a velocity $v = 20$ km/h. The bicycle wheel's radius is $r_{\text{W}} = 31$ cm. The radius of the dynamo wheel is $r_{\text{D}} = 0.5$ cm. Calculate the frequency f that is relevant for induction and compare to the value of a).

Solution 1.

a) From Ampere's law including the displacement current, we find in the gap of a capacitor (see Fig. 2) $\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$. The electric flux Φ_E is given by $E\pi R_{path}^2$ if the radius of the Amperian loop R_{path} is smaller than the capacitor radius R_0, and $E\pi R_0^2$ if the radius of the Amperian loop is larger than R_0. For $R_{path} < R_0$ we obtain:

$$B(2\pi R_{path}) = \mu_0 \epsilon_0 \pi R_{path}^2 \left(\frac{dE}{dt}\right) = \mu_0 \epsilon_0 \pi R_{path}^2 \left(\frac{1}{d} \frac{dV}{dt}\right)$$

$$\Rightarrow B = \mu_0 \epsilon_0 \pi R_{path}^2 \left(\frac{1}{d} \frac{2\pi f V_0}{2\pi R_{path}} \cos(2\pi f t)\right) = \mu_0 \epsilon_0 \pi R_{path}^2 \left(\frac{f V_0}{R_{path} d} \cos(2\pi f t)\right).$$

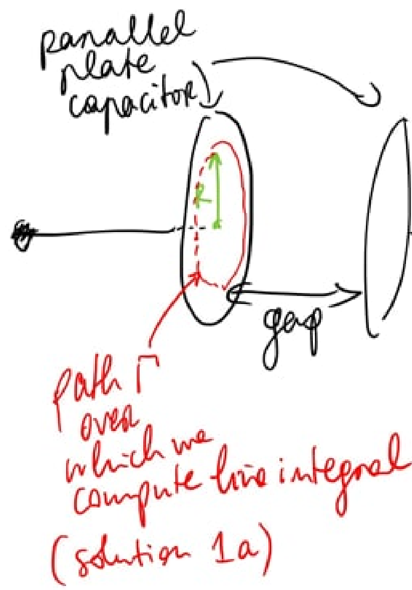


Figure 2: Sketch showing parallel plate capacitor and chosen line path for Ampere's law.

$\Rightarrow B_0(R) = \mu_0\epsilon_0\pi R_{path}\left(\frac{fV_0}{d}\right)$. Inside the capacitor, the field increases with the distance from the axis. For a distance R_{path} larger than R_0 from the central axis we have $B_0(R) = \mu_0\epsilon_0\pi R_0^2\left(\frac{fV_0}{R_{path}d}\right)$

b) For $R = R_{path} \leq R_0$ we have $B_0(R \leq R_0) = \mu_0\epsilon_0\pi R\left(\frac{fV_0}{d}\right)$ (See Fig. 3). Inserting the known numerical values one finds $B_0(R \leq R_0) = cR$ with $c = \mu_0\epsilon_0\pi\left(\frac{fV_0}{d}\right) \approx 6.29 \cdot 10^{-11}$ T/m.

For $R \geq R_0$ we have $B_0(R \geq R_0) = \mu_0\epsilon_0\pi R_0^2\left(\frac{fV_0}{Rd}\right) \rightarrow B_0(R \geq R_0) = c'\frac{1}{R}$ with $c' \approx 5.66 \cdot 10^{-14}$ Tm.

c) See Fig. 4.

d) We start from the definition of the displacement current density in vacuum $\vec{J}_D = \epsilon_0\frac{\partial\vec{E}}{\partial t}$. The flux of $\vec{J}_D$ over the surface $\Sigma = \pi R^2$ is the displacement current and it is equal to:

$\int_\Sigma \vec{J}_D \cdot d\vec{a} = I_D = \epsilon_0\int_\Sigma \frac{\partial\vec{E}}{\partial t} \cdot d\vec{a} = \frac{\partial}{\partial t}\epsilon_0\int_\Sigma \vec{E} \cdot d\vec{a} = \frac{\partial}{\partial t}\epsilon_0\Phi_E = \epsilon_0\frac{\partial\Phi_E}{\partial t} = \epsilon_0\frac{\pi R^2}{d}\frac{dV(t)}{dt} = \frac{2\pi^2\epsilon_0 V_0 f R^2}{d}cos(2\pi ft)$. To summarize:

$I_D(R,t) = \frac{2\pi^2\epsilon_0 V_0 f R^2}{d}cos(2\pi ft)$

The space-dependent component of I_D is maximum at $R = R_0$: $I_{D,max} = \frac{2\pi^2\epsilon_0 V_0 f R_0^2}{d}$.

Now we assume that the same amount of current flows through an infinitely long wire with diameter β. The generated magnetic field $\vec{B}_W$ is circumferential. Biot-Savart's law is used to compute the magnetic field generated by the current carrying wire at a distance $r = \beta/2$ with $\beta = 2$ mm:
$\vec{B}_W(\beta/2) = \hat{e}_\phi\frac{\mu_0 I_{D,max}}{2\pi(\beta/2)}$.

Therefore $B_W = \frac{2\pi\cdot(4\pi 10^{-7})\cdot(8.85\cdot 10^{-12})150\cdot 60\cdot 0.03^2}{0.005\cdot 0.002}T = 5.66 \cdot 10^{-11}$ T $= 56.6$ pT.

The magnetic field generated from the displacement current in the capacitor for the numerical values specified from the text amount to:

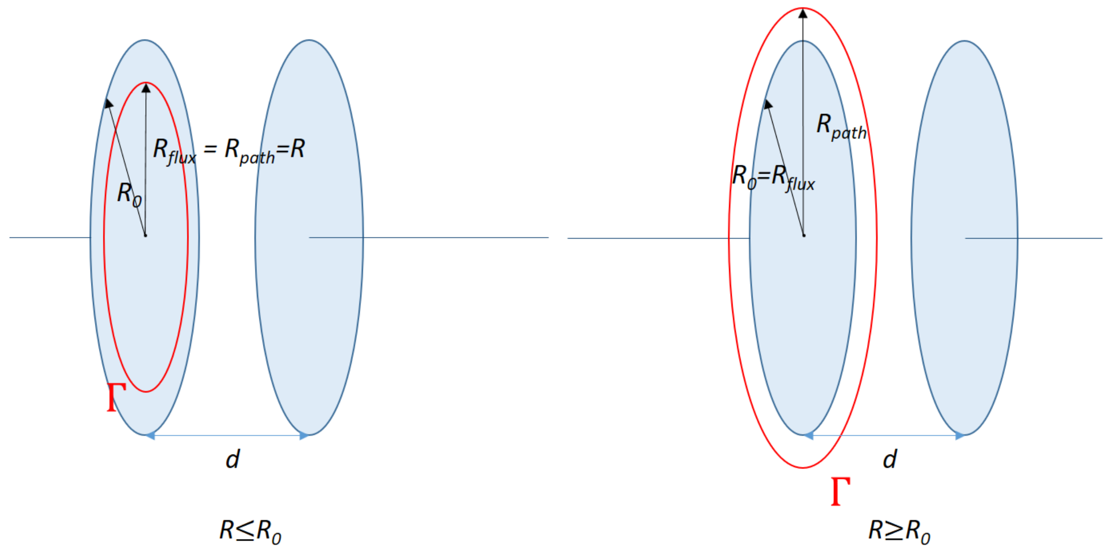


Figure 3: Sketch showing parallel plate capacitor and chosen line path for Ampere's law in the two cases: (a) $R \leq R_0$, and (b) $R \geq R_0$.

$B_C(R = 1\,\text{mm}) = \frac{\pi\epsilon_0\mu_0 V_0 f R}{d} = \frac{\pi\cdot(4\pi 10^{-7})\cdot(8.85\cdot 10^{-12})150\cdot 60\cdot 0.001}{0.005} T = 6.29 \cdot 10^{-14}\,\text{T} = 0.0629\,\text{pT}$

We conclude that B_C is smaller than B_W by a factor $B_C/B_W \approx 0.0011$. The magnetic field contributions due to $\frac{\partial E}{\partial t}$ are both absolutely and relatively very small. This justifies that such effects were neglected in the discussion of low frequency AC circuits in Chapter 6.

Solution 2.

a) We evaluate Ampere's law in a central $x-y$ plane. The magnetic field inside the coil at $a < r < b$ in the central $x-y$ plane can be calculated as $\oint \vec{B}\cdot d\vec{l} = 2\pi r B = \mu_0 N I \rightarrow \vec{B} = \dfrac{\mu_0 N I}{2\pi r}\hat{e}_\phi$. This solution is valid in different planes along z. The flux passing through one turn of the coil is the integral of this field over a cross-section of the coil in the $y-z$ plane i.e. $\Phi_1 = \int \vec{B}\cdot\hat{n}\,da = \int_a^b (B\hat{e}_\phi)\cdot(h\,dr\hat{e}_\phi) = \int_a^b \dfrac{\mu_0 N I}{2\pi r} h\,dr = \dfrac{\mu_0 N I h}{2\pi} ln\left(\frac{b}{a}\right)$. The total flux passing through the circuit of N turns is, $\Phi_{tot} = N\Phi_1 = \dfrac{\mu_0 N^2 I h}{2\pi} ln\left(\frac{b}{a}\right) \equiv LI$. The self-inductance of this coil is $L = \dfrac{\mu_0 N^2 h}{2\pi} ln\left(\frac{b}{a}\right)$.

b) Since there is no magnetic core, the magnetic energy density is given by $u = \dfrac{1}{2}\dfrac{B^2}{\mu_0} = \dfrac{\mu_0 N^2 I^2}{8\pi^2 r^2}$.

c) The total energy U is the integration of energy density u over the volume of the coil i.e. $U = \int_V dV\,u = \int_a^b 2\pi r h\,dr\left(\frac{\mu_0 N^2 I^2}{8\pi^2 r^2}\right) = \dfrac{\mu_0 N^2 I^2 h}{4\pi} ln\left(\frac{b}{a}\right)$. To check the result we calculate U/L with L calculated in part (a) which reads $\frac{U}{L} = \frac{1}{2}I^2$. Hence, we find $U = \frac{1}{2}LI^2$ by rearranging the terms. This is the general formula for energy stored in an inductance. The result of (c) is consistent with (a).

d) $L = \dfrac{4\pi\times 10^{-7}\times 10^4\times 5\times 10^{-3}}{2\pi} ln\left(\frac{7.5}{5}\right)\,\text{H} = 4.05\,\mu\text{H}$. For energy stored in the coil we have $U = \frac{1}{2}LI^2 = 0.5\times 4.1\times 10^{-6}\times 9\ \text{J} \simeq 18.25\,\mu\text{J}$. The capacitance is $C = \dfrac{\epsilon_r\epsilon_0(\pi b^2)}{d} = 1.56\text{pF}$. Therefore, the required potential difference amounts to $V = \sqrt{\dfrac{2U}{C}} = 4830.5$ V.

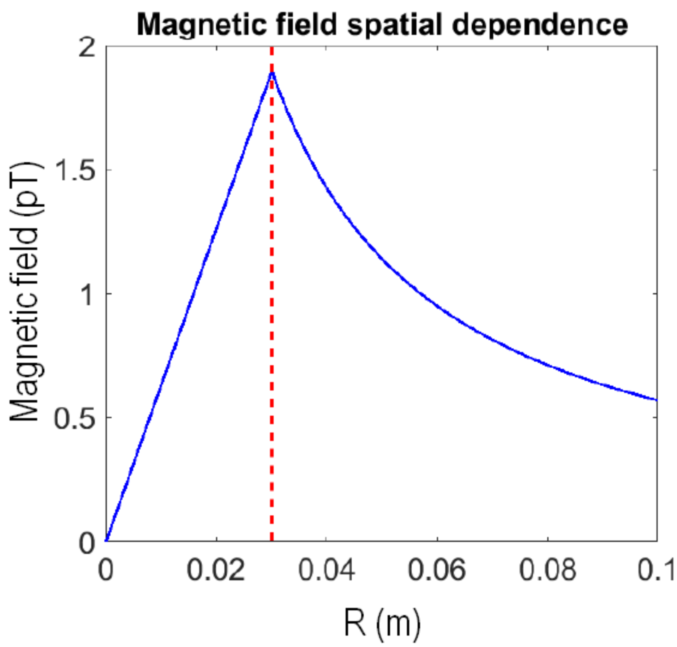


Figure 4: Magnetic field is continuous at the transition $R = R_0$

Solution 3.
We exploit the disk symmetry and apply cylindrical coordinate (r, z, φ).

a) The velocity of a generic point P within disk's radius is $\vec{v}_r = \omega r\hat{\varphi}$, with r being the distance from the disk's axis. Assuming a charged particle with charge q at point P the magnetic Force $\vec{F}_{\mathrm{M}}$ acting on the particle is:
$\vec{F}_{\mathrm{M}}(r) = q\vec{v}_r \times \vec{B} = q(\omega r\hat{\varphi}) \times (B\hat{z}) = q\omega Br\hat{r}$

The force has radial direction.

For the following calculation it is important to realize that in equilibrium without a connected circuit one finds for the Lorentz force $\vec{F}_{\mathrm{L}} = q\vec{E}(r) + q\vec{v}_r \times (B\hat{z}) = 0$. This equation indicates that $\vec{v}_r \times (B\hat{z})$ corresponds to an electric field.

From here we find the expression of $\vec{E}(r) = \frac{q\vec{v}_r \times (B\hat{z})}{q} = \frac{\vec{F}_{\mathrm{M}}(r)}{q}$ for the electric field originating from the magnetic force $\vec{F}_{\mathrm{M}}$. Hence $\vec{F}_{\mathrm{M}}(r)$ divided by the charge q is *equivalent* to the local electric field $\vec{E}(r)$ acting on the charge. This electric field gives rise to the electromotive force ΔV between the center with $r = 0$ and the outer edge with $r = a$ of the disc. To compute ΔV we evaluate the electrostatic potential difference due to the assumed electric field $F_{\mathrm{M}}(r)/q$ between the center of the disk and a point on the outer edge:
$\Delta V = \int_0^a \frac{1}{q}\vec{F}_{\mathrm{M}} \cdot d\vec{r} = \int_0^a \omega Br\hat{r} \cdot d\vec{r} = \int_0^a \omega Br \cdot dr = \frac{\omega Ba^2}{2}$.

Using Ohm's law we find for the current I flowing in the circuit:
$I = \frac{\Delta V}{R} = \frac{\omega Ba^2}{2R}$.

b) The dissipated power P_{diss} through the resistor R caused by the electromotive force ΔV is
$P_{diss} = \frac{(\Delta V)^2}{R} \Rightarrow \Delta V = \sqrt{P_{diss}R} \Rightarrow \sqrt{P_{diss}R} = \frac{\omega Ba^2}{2} \Rightarrow \omega = \frac{2}{Ba^2}\sqrt{\Delta P_{diss}R}$.

Inserting numerical values it is found that
$\frac{1}{2\pi}\omega = \frac{1}{2\pi}5 \cdot 10^4 \frac{\mathrm{rad}}{s} = 7.96\,\mathrm{kHz}$.

The device can not be used as a bicycle dynamo.

Solution 4.

a) We write down the formula for the magnetic flux induced in the coil $\Phi(t) = B(t)\pi a^2 N$ from which we obtain the emf $emf(t) = -\frac{d\Phi}{dt} = -B_0 \cos(\omega t)\omega\pi a^2 N$.

In the following we motivate the formula for the time-averaged power $P_{\text{avg}} = emf_0^2/2R$ given in the problem, where emf_0 is the amplitude of the time-dependent electromotive force. We start by stating the time-dependent power dissipated in the load resistor $P(t) = emf^2(t)/R$ using the known formula for power in a resistor. This value is now averaged over one period of rotation of the dynamo:

$$P_{\text{avg}} = \langle P \rangle = \frac{1}{R}\langle(\mathcal{E}_{emf})^2\rangle = \frac{1}{R}(B_0\omega\pi a^2 N)^2 \frac{1}{T}\int_0^T \cos^2(\omega t)dt$$

The brackets $\langle\rangle$ represent the mean value (averaged value) over a period T, that's why we consider the integral ranging from 0 to T of the time dependent term $(\cos(\omega t))^2$ and then divide the result of the integration by one period T.

The integration over time results in

$$\begin{aligned}
\frac{1}{T}\int_0^T \cos^2(\omega t)dt &= \frac{1}{T}\int_0^T \left(\frac{1 + cos(2\omega t)}{2}\right) dt \\
&= \frac{1}{2T}\int_0^T dt + \frac{1}{2T}\int_0^T \cos(2\omega t)dt \\
&= \frac{1}{2} + \frac{1}{2T}\left[\frac{\sin(2\omega t)}{2\omega}\right]_0^T \\
&= \frac{1}{2} + \frac{1}{2T}\left[\frac{\sin\left(2 \times \frac{2\pi}{T} \times T\right) - \sin(0)}{2 \times \frac{2\pi}{T}}\right] = \frac{1}{2}.
\end{aligned}$$

One concludes that
$\langle P \rangle = \frac{1}{R}\langle(\mathcal{E}_{emf})^2\rangle$ in general. This relation is now used to find the solution of the problem:
$\langle P \rangle = \frac{1}{2}\frac{1}{R}(B_0\omega\pi a^2 N)^2$.

Therefore, the relation between the frequency of the rotation and the average power is obtained by inverting the previous equation and isolating and solving for ω:

$$\sqrt{2\langle P \rangle R} = B_0 N a^2 \pi\omega \Rightarrow \omega = \frac{\sqrt{2\langle P \rangle R}}{B_0 N a^2 \pi}.$$

Inserting numerical values one finds $\frac{1}{2\pi}\omega = \frac{1}{2\pi}\frac{\sqrt{2\cdot 4\cdot 1}}{0.2\cdot 1000\cdot 0.02^2\pi}$ Hz $= 1.79$ Hz.

b) One turn of the cycle wheel corresponds to $\frac{r_w}{r_D} = 62$ turns of the dynamo. At $v = 20\,\text{km/h} = 5.56\,\text{m/s}$ the wheel rotates with a frequency $f_w = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{5.56\,\text{m/s}}{0.31\,\text{m}} = 2.85$ Hz.

This means that the induction occurs at a frequency $f = 62 f_w = 176.7$ Hz. This value is well larger than the value found in (a). Hence at a much smaller velocity of the bike than 20 km/h the lamp is already bright and helps one to find the way in the dark.