Exercise sheet 10: Induction, inductances, LC oscillator

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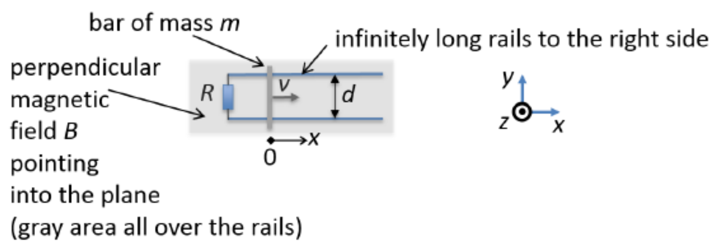
20/11/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.
(Induction and motion as considered in General Physics I /Category II)
A metal bar of mass m slides with velocity v and without friction on two parallel conducting rails (see figure). The rails are semi-infinite and separated by a distance d. At one end of the rails a resistor with resistance R electrically connects the rails. Assume that the resistances of bar and rails are negligible. A uniform field B is perpendicular to the plane and points along the $-\vec{z}$-direction.

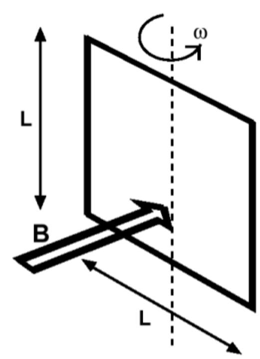


a) Calculate the current I flowing in the circuit. How does it depend on v, B, R and geometrical parameters (neglect the self-inductance)? Indicate the direction of the current when velocity v is pointing to the right.

b) Assume now that at a given position $x = 0$ and time $t = 0$, the bar has an initial velocity v_0 parallel to the rails, pointing away from the resistor. No external force is applied to the bar for $t > 0$. Calculate the formula for the time-dependent velocity $v(t)$ of the bar.

Exercise 2.
(Rotating square loop / Category II
(after training time needed: 20-25 min))

Consider a square loop rotating around a vertical axis with constant angular velocity ω. The square loop is made of a wire of resistance R. The loop is placed in a uniform magnetic field $\vec{B}$ that is perpendicular to the axis of the rotation, see figure.



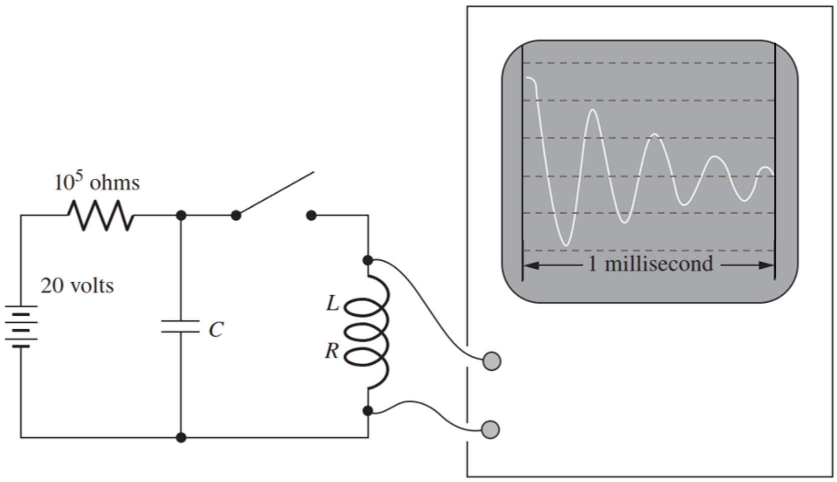
a) Calculate the induced current I in the loop. Evaluate the current flow direction for the sketched graph.

b) Calculate the torque τ necessary to keep the loop turning.

c) Show that the time-dependent mechanical power $P_{mech}(t) = \tau(t)\omega$ (General Physics I) is equal to the electrical power dissipated in the wire.

Exercise 3.
(Decaying signal/Category I)

The coil in the circuit of the figure below is shown to have an inductance of 0.01 H. When the switch is closed, the oscilloscope sweep is triggered. The 10^5 ohm resistor is large enough (as you will discover) so that it can be treated as essentially infinite for part (a) and (b) of this problem.

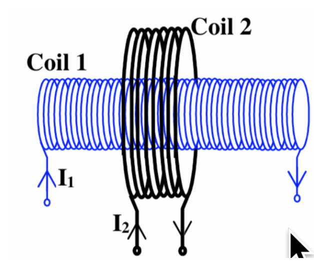


a) Determine, as well as you can, the value of the capacitance C neglecting first any resistance in the loop formed by the coil and capacitor.

b) Now estimate the value of the resistance R of the coil. We assume that only this value R is responsible for the observed decay of the oscillating signal. Hint: Consider R to be in series with L.

c) What is the magnitude of the voltage across the oscilloscope input, a“long” time after the switch has been closed, say several seconds after?

Exercise 4.
(Self-inductance versus mutual inductance/Category II/
(After training for solution: 30 min))



The current I_1 through a long solenoid (coil 1) with n_1 turns per meter and radius R_1 is changing with time (given by dI_1/dt). A further circular coil 2 surrounds coil 1 (see figure). For the calculation of the magnetic field in coil 1 assume the solenoid to be infinitely long.

a) First disregard coil 2. Calculate the strength of the electric field induced by dI_1/dt as a function of distance r from the central axis of the solenoid. Sketch the result as a function of r.

b) Use the previous approximation: (i) Calculate the maximum emf developed along the solenoid assuming the following parameters: $R_1 = 1$ cm, length $l_1 = 15$ cm, and an absolute number of turns $N_1 = 150$. Here, l_1 is the length of the considered segment. The current follows $I_1(t) = I_0 sin(\omega t)$ with $I_0 = 5$ A and $\omega = 2\pi 50$ Hz. (ii) Are the maximum I and maximum emf reached at the same time? (Hint: The field lines are encircled N_1 times by the wire of the solenoid)

c) Now consider coil 2 to be present. It has a total of N_2 turns. What is the mutual inductance M when one assumes that all the flux from the solenoid (coil 1) passes through the outer coil 2; how does it depend on the parameters n_1, N_2, and R_1? (Hint: No current flow in coil 2, ends of coil 2 are open. To calculate M one must consider that coil 2 encircles the flux of coil 1 N_2 times.)

d) Relate the mutual inductance M to the self-inductances L_1 and L_2 of the solenoid (coil 1) and the coil 2, respectively. Assume that coil 2 is wound very tightly onto the solenoid ($R_2 = R_1$), and it has the same length l. Still, the number of its turns is different. The approximation of the magnetic field of an infinitely long solenoid is considered to be valid in both cases.

Solution 1.

a) The force acting on the system is the magnetic force $F = q|\vec{v} \times \vec{B}| = q|\vec{E}|$. The term $|\vec{v} \times \vec{B}|$ is equivalent to an electric field $|\vec{E}| = |\vec{v} \times \vec{B}|$.

Because of this there is a voltage drop ΔV between the conducting rails: $\Delta V = |\vec{E}|d$. From Ohm's law it follows that a current I flows through the system that is $I = \frac{\Delta V}{R}$.

From these considerations one obtains that $I = \frac{\Delta V}{R} = \frac{|\vec{E}|d}{R} = \frac{|\vec{v}\times\vec{B}|d}{R}$. The bar is moving perpendicularly to the field. One concludes that:
$I = \frac{vBd}{R}$

The bar moves to the right. This causes a flux increase because the area is increasing. The induced current flows counterclockwise to counteract the flux increase.

Another approach to find the current I is to use the expression of the electromotive force $emf = -\frac{d\phi}{dt} = -\frac{d(BA)}{dt} = -\frac{d(B\cdot d\cdot v\cdot t)}{dt} = -Bdv$. Once the emf is estimated then using Ohm's law one finds $I = emf/R = \frac{-Bdv}{R}$

b) One notices that this is a 1D problem: Newton's law yields $F = IdB = -\frac{Bdv}{R}dB = m\frac{dv}{dt} \Rightarrow -\frac{v(Bd)^2}{R} = m\frac{dv}{dt}$. From this one should separate the variables and solve for the velocity v:

$\frac{dv}{v} = -\frac{(Bd)^2dt}{mR} = -\frac{dt}{\tau}$. We have defined $\frac{1}{\tau} = \frac{(Bd)^2}{mR}$.

$\int_{v(0)}^{v(t')} \frac{dv}{v} = -\int_0^{t'} \frac{dt}{\tau}$

$ln[v(t')] - ln[v(0)] = -\frac{t'}{\tau}$. We rename $v(0) = v_0$.

$v(t') = v_0 e^{-t'/\tau}$: this is the time-dependent expression for the velocity.

Solution 2.

a) The magnetic flux Φ in the wire is given by $\Phi = BL^2 cos(\theta)$ where θ is defined in Fig. 1 (The specific definition of θ is helpful for the calculation of the torque shown below). The angle θ depends on the time t following $\theta = \omega \cdot t + \theta_0$, where θ_0 is the angle at $t = 0$. The induced current in the loop is then given by $I = \frac{\Delta V}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}\frac{d}{dt}(BL^2\cos(\omega t + \theta_0)) = \frac{BL^2}{R}\omega\sin(\omega t + \theta_0)$ using Ohm's law $I = \frac{\Delta V}{R}$ (where ΔV is the voltage induced by Henry-Faraday induction law due to the rotation of the loop).

b) The magnetic force acting onto the induced current I calculated is $F = ILB$ because field B and current I are along directions which are perpendicular to each other. The corresponding torque (which one needs to counteract to rotate the loop at constant angular velocity) is generated by a pair of forces acting on the two sides of the loop, i.e.,

$\vec{\tau} = \vec{r}\times\vec{F} + (-\vec{r}\times(-\vec{F})) = 2\vec{r}\times\vec{F}$ (Fig. 1) where r is a vector pointing from the rotation axis to the relevant point where we consider the force (the length of r is $L/2$). The torque applied to the loop for constant angular velocity is $-\vec{\tau}$ such that the sum of both torques is zero (Newton's first law for constant velocity). One finds for the absolute value of the torque $|\vec{\tau}| = 2F\frac{L}{2}\sin(\theta) = \frac{(BL^2)^2\omega}{R}\sin^2(\omega t + \theta_0)$. The torque is time-dependent.

c) The mechanical power is $P_{mech}(t) = \tau(t)\omega$ and the electrical power is $P_{el}(t) = RI^2(t)$. Inserting the previous results in the equations one discovers that $P_{mech}(t) = \tau(t)\omega = RI^2(t) = P_{el}(t)$.

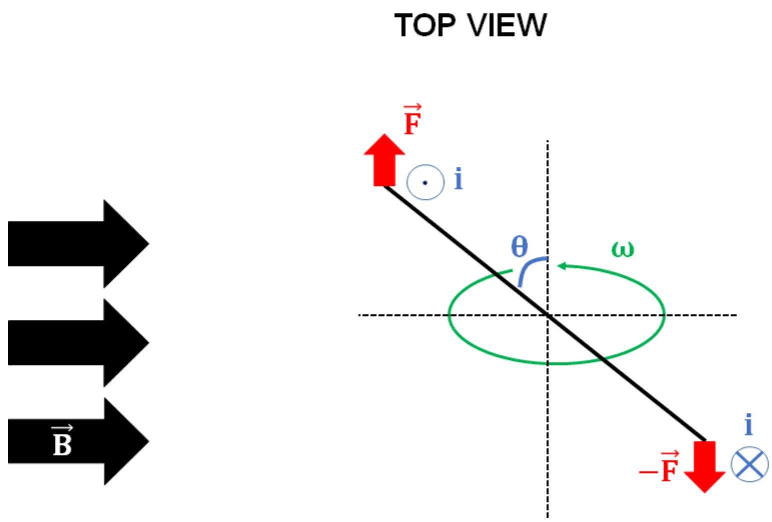


Figure 1: Sketch to understand the torque within the system. Magnetic field is incoming from the left, force acting on the sides of the loop has different directions as the induced current in those regions flows in opposite directions. The rotation of the loop is assumed to be counter-clockwise.

Solution 3.

a) We consider only the right part of the entire circuit, this part forms an RLC circuit. Left part is not considered as its resistance $r = 10^5\,\Omega$ is much larger than the resistance R attributed to the coil. By Kirchhoff's law we know $L\frac{dI}{dt} + RI + \frac{1}{C}Q = 0$ under the following assumption: we assume the current I to flow to the upper capacitor plate which is still charging up and already positively charged a bit. Differentiating with respect to time this equation yields the RLC circuit difference equation $L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = 0$.

The RLC circuit differential equation is rewritten dividing all terms by the inductance L:

$\frac{d^2I(t)}{dt^2} + \frac{R}{L}\frac{dI(t)}{dt} + \frac{1}{LC}I(t) = 0.$

A solution to this equation is $I(t) = I_{max}\exp(-\gamma t)\sin(\omega t + \alpha)$ with damping constant $\gamma = \frac{R}{2L}$ and $\omega = \sqrt{\omega_0^2 - \gamma^2}$. Assuming a low damping constant one obtain $\omega = \omega_0$. In the plot four cycles are shown to be completed in $\Delta t = 1$ ms. Therefore $\omega = 2\pi\frac{4}{10^{-3}} = 25133$ rad/s. Assuming then low damping one finds $C = \frac{1}{L\omega^2} = \frac{1}{(25133\,\mathrm{rad/s})^2 0.01\,\mathrm{H}} = 1.58 \times 10^{-7}$ F.

b) For $t = \gamma^{-1}$ the signal has been reduced by $1/e$. This occurs approximately for $t = 0.5$ ms. We attribute the decay to the dissipation in the resistance of the coil, hence $R = R_L$ in a). Therefore $\gamma = \frac{R_L}{2L} = \frac{1}{0.5\,\mathrm{ms}} \Rightarrow R_L = \frac{2L}{0.5\,\mathrm{ms}} = \frac{2\cdot 10\,\mathrm{mH}}{0.5\,\mathrm{ms}} = 40.0\,\Omega$

c) After a long time $T \gg \gamma^{-1}$ oscillations have decayed and only DC current flows in the circuit through the two resistors in series $r = 100\,\mathrm{k}\Omega$ and $R_L = 40\,\Omega$. We conclude $V_{battery} = V_r + V_{R_L} \Rightarrow V_{R_L} = V_{battery} - rI = V_{battery} - r\frac{V_{battery}}{r+R_L} = \frac{R_L}{r+R_L}V_{battery} = 7.997$ mV.

Solution 4.

a) Let us use Maxwell's law (integral form of induction's law) to relate electric and magnetic field:

$\oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{d\Phi_{enc}}{dt}.$

where Φ_{enc} is the enclosed magnetic flux. Here we assume an infinitely long solenoid therefore its magnetic field inside is given by

$B = \mu_0 n_1 I_1,$

where n_1 represents the number of turns (windings) per meter of coil 1. Firstly let us consider a closed loop inside the solenoid

$r < R_1$, then

we have

$E 2\pi r = -\pi r^2 \frac{dB}{dt} \Rightarrow E = -\frac{r}{2}\mu_0 n_1 \frac{dI_1}{dt}$.

Inside the solenoid the absolute value (magnitude $\mid E \mid$) of the electric field increases linearly with the radial distance r (Fig. 2).

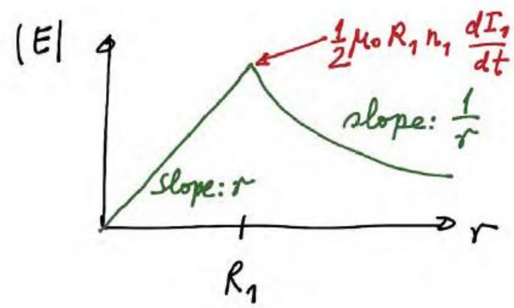


Figure 2: Sketch displaying the magnitude and dependence of $\mid E \mid$ as a function of distance r from the center of the infinitely long coil.

Secondly let us focus on a closed loop outside the solenoid i.e $r > R_1$ so that we find

$E 2\pi r = -\pi R_1^2 \frac{dB}{dt} \Rightarrow E = -\frac{R_1^2}{2r}\mu_0 n_1 \frac{dI_1}{dt}$.

Outside the solenoid the absolute value of the electric field decreases as r^{-1}. At $r = R_1$ the field is

$E(r = R_1) = -\frac{R_1}{2}\mu_0 n_1 \frac{dI_1}{dt}$.

b) $emf = \Delta V = -\frac{d\Phi}{dt}$.

The flux Φ in consideration is the one through <u>all</u> the turns of coil 1, i.e., $N_1 = n_1 \cdot l_1$. Therefore

$\Delta V(t) = -N_1 \frac{d\Phi_{single\,turn}}{dt} = -N_1 \mu_0 n_1 \pi R_1^2 \frac{dI_1}{dt} = -N_1 \mu_0 n_1 \pi R_1^2 I_0 \omega \cos(\omega t)$.

We conclude

$\mid \Delta V_{max} \mid = N_1 \mu_0 n_1 \pi R_1^2 I_0 \omega = 150 \cdot 4\pi 10^{-7} (\mathrm{J}/(\mathrm{A}^2\mathrm{m})) \frac{150}{15\,\mathrm{cm}} \pi (1\,\mathrm{cm})^2 \cdot 5\mathrm{A} \cdot 2\pi 50\mathrm{Hz} = 0.093$ V.

The maximum of the electromotive force and that of the current are not reached at the same time. Indeed when $\Delta V = \Delta V_{max}$ the current is zero according to the proposed temporal dependence.

c) For this calculation we need to calculate the flux of the magnetic field of coil 1 through the cross-sectional area of coil 2. The induction law reads $\Delta V_2 = -\frac{d\phi_{21}}{dt}$,

where $\Phi_{21} = M \cdot I_1$ is the magnetic flux through coil 2 due to a current I_1 in coil 1. M is the mutual inductance. Following the problem set the flux in coil 2 is considered to be the flux of coil 1. The relevant flux is hence calculated from the magnetic field generated by coil 1 ($B_1 = \mu_0 n_1 I_1$) times πR_1^2:

$\Delta V_2 = -N_2 \mu_0 n_1 \pi R_1^2 \frac{dI_1}{dt}$

The factor N_2 is needed as coil 2 encircles the magnetic field lines of coil 1 N_2 times. We find that the mutual inductance M is

$M = N_2 \mu_0 n_1 \pi R_1^2$.

d) Making use of the assumptions given by the text and rearranging terms in the solution of item (c) we obtain:

$M = N_2 \mu_0 n_1 \pi R_1^2 = N_2(\mu_0 n_1 \pi R^2) = (\mu_0 n_2 \pi R^2)(\mu_0 n_1 \pi R^2)\frac{l}{\mu_0 \pi R^2} = (\mu_0 n_2^2 \pi R^2)(\mu_0 n_1^2 \pi R^2)\frac{l}{\mu_0 \pi R^2 n_1 n_2} = (\mu_0 n_2^2 l \pi R^2)(\mu_0 n_1^2 l \pi R^2)\frac{l}{\mu_0 \pi R^2 n_1 n_2 l^2} = L_1 L_2 \frac{1}{\mu_0 \pi R^2 n_1 N_2} = \frac{L_1 L_2}{M} \Rightarrow M^2 = L_1 L_2 \Leftrightarrow M = \sqrt{L_1 L_2}$, with L_1 and L_2 being the self-inductances of coil 1 and coil 2.