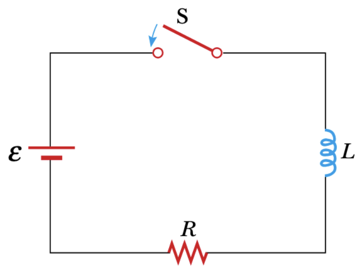
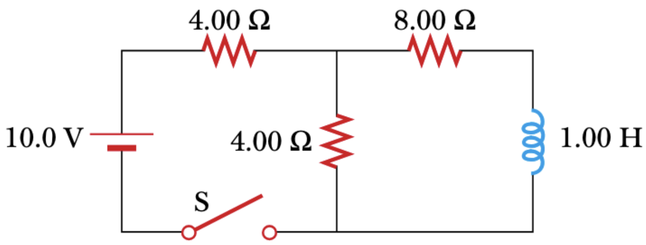
Exercise sheet #12

Problem 1. In the circuit shown below, let $L = 7.00\text{H}, R = 9.00\Omega$, and $\varepsilon = 120$ V. What is the self-induced emf 0.200 s after the switch is closed?

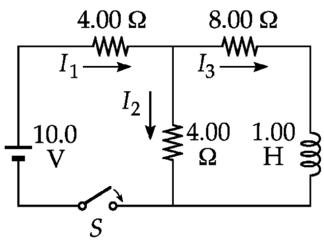


Solution: This circuit has only one loop. From Kirchhoffs rule we get only one (differential) equation $\varepsilon - L\frac{dI}{dt} - IR = 0$. We can separate the variables, $\frac{dI}{\varepsilon - IR} = \frac{dt}{L}$, and integrate both sides of the equation (within appropriate limits). With the reference for time at the instant of closing the circuit $\int_0^{I(t)} \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L}$. (Note that we used one symbol t for two different quantities!). From the fundamental theorem of calculus $\ln \frac{\varepsilon - I(t)R}{\varepsilon} = -\frac{R}{L}t$. The solution is a time dependent function $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$. We can rewrite this equation as $I = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$. Since $\tau = L/R = 0.78$ s, we have $I = \frac{120 \text{ V}}{9.00\Omega}\left(1 - e^{-0.26}\right) = 3.05$ A. Now $\Delta V_R = IR = 27.4$ V and $\Delta V_L = \varepsilon - \Delta V_R = 92.6$ V. □

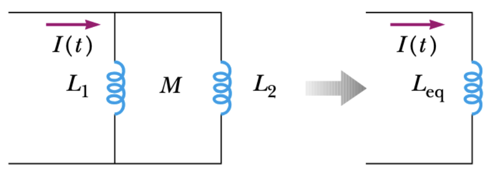
Problem 2. The switch in the figure below is open for $t < 0$ and then closed at time $t = 0$. Find the current in the inductor and the current in the switch as functions of time thereafter.



Solution: Naming the currents as shown in the figure below and aplying Kirchhoff's laws we obtain $I_1 = I_2 + I_3$, $10.0 \text{ V} - 4.00I_1 - 4.00I_2 = 0$, and $10.0 \text{ V} - 4.00I_1 - 8.00I_3 - 1.00\frac{dI_3}{dt} = 0$. From the first two equations it follows that $10.0 \text{ V} + 4.00I_3 - 8.00I_1 = 0$ and $I_1 = 0.50I_+3 + 1.25$ A. Then the last equation can be rewritten as $10.0 \text{ V} - 4.00\left(0.500I_3 + 1.25 \text{ A}\right) - 8.00I_3 - 1.00\text{H}\frac{dI_3}{dt} = 0$, yielding $1\text{H}\frac{dI_3}{dt} + 10.0\Omega I_3 = 5.00$ V. We solve the differential equation to obtain $I_3(t) = \frac{5.00 \text{ V}}{10.0\Omega}\left[1 - e^{-10.0\Omega t/1.00\text{H}}\right] = 0.50 \text{ A}\left[1 - e^{-10t/s}\right]$. Then $I_1 = 1.25 + 0.50I_3 = 1.50 \text{ A} - 0.25 \text{ A}e^{-10t/s}$. □



Problem 3. Two inductors having self-inductances L_1 and L_2 are connected in parallel as shown in the figure below. The mutual inductance between the two inductors is M. Determine the equivalent selfinductance L_{eq} for the system.



Solution: With $I = I_1 + I_2$, the voltage across the pair is: $\Delta V = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} = -L_{eq}\frac{dI}{dt}$. Hence, $-\frac{dI_1}{dt} = \frac{\Delta V}{L_1} + \frac{M}{L_1}\frac{dI_2}{dt}$ and $-L_2\frac{dI_2}{dt} + \frac{M\Delta V}{L_1} + \frac{M^2}{L_1}\frac{dI_2}{dt} = \Delta V$, yiedling

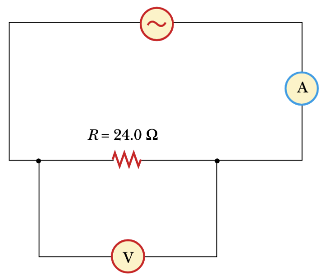
$$\left(-L_1L_2 + M^2\right)\frac{dI_2}{dt} = \Delta V\left(L_1 - M\right)$$

By substitution, $-\frac{dI_2}{dt} = \frac{\Delta V}{L_2} + \frac{M}{L_2}\frac{dI_1}{dt}$ leads to

$$\left(-L_1L_2 + M^2\right)\frac{dI_1}{dt} = \Delta V\left(L_2 - M\right)$$

Adding the last two equations we get: $\left(-L_1L_2 + M^2\right)\frac{dI}{dt} = \Delta V\left(L_1 + L_2 - 2M\right)$, and therefore $L_{eq} = -\frac{\Delta V}{dI/dt} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$. □

Problem 4. An AC power supply produces a maximum voltage of $V_0 = 100$ V. This power supply is connected to a $24.0 - \Omega$ resistor, and the current and resistor voltage are measured with an ideal AC ammeter and an ideal AC voltmeter, as shown below. What does each meter read? Recall that an ideal ammeter has zero resistance and an ideal voltmeter has infinite resistance.

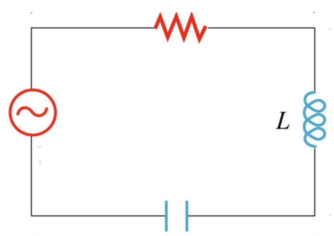


Solution: The meters measure the rms values of potential difference and current. These are $V_{rms} = \frac{V_0}{\sqrt{2}} = 70.7$ V and $I_{rms} = \frac{V_{rms}}{R} = 2.95$ A. □

Problem 5. (a) For the series RLC connection shown below, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors.

(b) An RLC circuit consists of a $150 - \Omega$ resistor, a $21 - \mu$F capacitor and a $460 -$ mH inductor, connected in series with a $120 -$ V, $60 -$ Hz power supply. What is the phase angle between the current and the applied voltage?

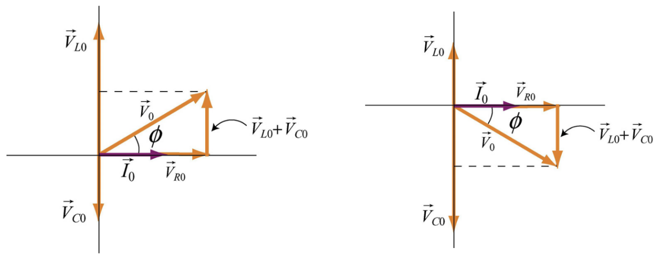
(c) Which reaches its maximum earlier, the current or the voltage?



Solution: (a) For the series connection, Yhe instantaneous voltage across the system is equal to the sum of voltage across each element. The phase angle between the voltage (across the system) and the current (through the system) is: $\phi = \arctan \frac{X_L - X_C}{R}$. In the figure below the phasor diagram for a series RLC circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$. On the one hand, in the inductive case, $V_{0,L} > V_{0,c}$, we see that $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. Aditionally, in the capacitance case, $V_{0,C} > V_{0,L}$, we have that $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

(b) From the definition, the inductive reactance of the inductor is $X_L = \omega L = 2\pi \cdot 60 \text{ Hz} \cdot 0.46\text{H} = 173\Omega$. From the definition, the capacitive reactance of the capacitor is $X_C = \frac{1}{\omega C} = \frac{1}{2\pi \cdot 60 \text{ Hz} \cdot 21\times 10^{-6} \text{ F}} = 126\Omega$. The phase angle between the voltage (across the system) and the current (through the system) is: $\phi = \arctan \frac{X_L - X_C}{R} = \arctan \frac{174\Omega - 126\Omega}{150\Omega} = 17.4°$

(c) The voltage leads the current.



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