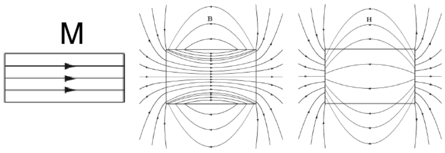
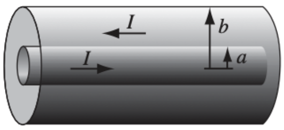
Exercise sheet #11

Problem 1. For a bar magnet (a short circular cylinder of radius a and length L with a "frozen-in" uniform magnetization M parallel to its axis), make careful sketches of $\mathbf{M}, \mathbf{B}$, and $\mathbf{H}$, assuming L is about $2a$. Compare with your results in Problem 3 from Exercise sheet 7.

Solution: $\mathbf{B}$ is the same as the field of a short solenoid; $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$. □



Problem 2. A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface (See figure below). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.



Proof. Using Ampere's law for $\mathbf{H}$, considering a loop of radius $a < s < b$ centered at the small cylinder:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} = I, \text{ so } \mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

$$\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu_0 (1 + \chi_m) \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

$$\mathbf{M} = \chi_m \mathbf{H} = \frac{\chi_m I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\chi_m I}{2\pi s}\right)\hat{\mathbf{z}} = 0$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \begin{cases} \frac{\chi_m I}{2\pi a}\hat{\mathbf{z}}, & \text{at } s = a \\ -\frac{\chi_m I}{2\pi b}\hat{\mathbf{z}}, & \text{at } s = b \end{cases}$$

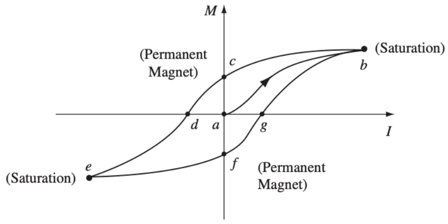
Total enclosed current, for an amperian loop between the cylinders:

$$I + \frac{\chi_m I}{2\pi a} 2\pi a = (1 + \chi_m) I, \quad \text{so} \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0 (1 + \chi_m) I \Rightarrow \mathbf{B} = \frac{\mu_0 (1 + \chi_m) I}{2\pi s}\hat{\boldsymbol{\phi}}. \checkmark$$

□

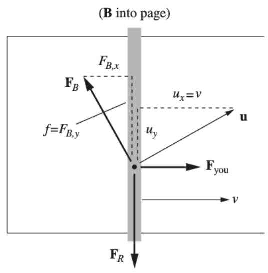
Problem 3. How would you go about demagnetizing a permanent magnet at point c in the hysteresis loop pictured below? That is, how could you restore it to its original state, with $M = 0$ at $I = 0$?

Solution: Place the object in a region of zero magnetic field, and heat it above the Curie point—or simply drop it on a hard surface. If it's delicate (a watch, say), place it between the poles of an electromagnet, and magnetize it back and forth many times; each time you reverse the direction, reduce the field slightly. □



Problem 4. In the figure below a conducting rod is pulled to the right at speed v while maintaining contact with two rails. A magnetic field points into the page. From the reasoning in lecture, we know that an induced emf will cause a current to flow in the counterclockwise direction around the loop. Now, the magnetic force $q\mathbf{uB}$ is perpendicular to the velocity $\mathbf{u}$ of the moving charges, so it can't do work on them. However, the magnetic force $\mathbf{f}$ in the equation $\mathcal{E} \equiv \frac{1}{q}\int \mathbf{f} \cdot d\mathbf{s}$ certainly looks like it is doing work. What's going on here? Is the magnetic force doing work or not? If not, then what is? There is definitely something doing work because the wire will heat up.

Proof. The figure below outlines the situation described:

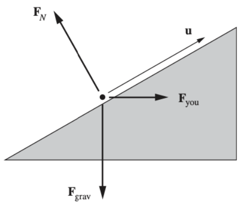


For simplicity, we assume that the mobile charges are positive; this doesn't affect the result. The important point to realize is that there are two components to a given charge's velocity $\mathbf{u}$, namely the horizontal component $u_x = v$ due to the motion of the rod, and the vertical component u_y due to the current along the rod. This means that the magnetic force $\mathbf{F}_B$ points up and to the left, perpendicular to $\mathbf{u}$, as shown. Its magnitude is $F_B = quB$, and its two components have magnitudes $F_{B,x} = qu_yB$ and $F_{B,y} = qu_xB = qvB$. The latter of these is what we called $\mathbf{f}$ in the equation for $\mathcal{E} \equiv \frac{1}{q}\int \mathbf{f} \cdot d\mathbf{s}$. Assuming that the current is steady and the charge isn't accelerating, the total force on it equals zero. So if you are applying the force to the rod, then your force is given by $F_{\text{you}} = F_{B,x}$, and the resistive force on the charges is given by $F_R = F_{B,y}$. (All of these quantities are magnitudes, so they are defined to be positive.)

Which forces do work? As mentioned in the problem, the magnetic force does no work because $\mathbf{F}_B$ is perpendicular to $\mathbf{u}$. But if you wish, you can break this zero work into two equal and opposite pieces. The vertical component of $\mathbf{F}_B$ does work at a rate $F_{B,y}u_y = (qu_xB)\,u_y$. And the horizontal component does work at a rate $-F_{B,x}u_x = -\left(qu_yB\right)u_x$. These two rates are equal and opposite, as they must be. You also do work, because there is a component of $\mathbf{u}$ in the direction in which you are pulling. The rate at which you do work is $F_{\text{you}}\,u_x$. And due to the balancing of all the forces, this positive rate is equal and opposite to the negative rate at which $F_{B,x}$ does work. The resistive force also does work, and the rate is $-F_Ru_y$. This negative rate is equal and opposite to the positive rate at which $F_{B,y}$ does work.

We see that the magnetic force does zero net work, while the positive work you do is canceled by the negative work the resistive force does. While it is true that a component of $\mathbf{F}_B$ does positive work (the vertical component, which we called $\mathbf{f}$), the other component of $\mathbf{F}_B$ does an equal and opposite amount of negative work. So it would hardly be accurate to say that the magnetic force does work.

This setup is essentially the same as the setup in which you push a block up a frictionless inclined plane at constant speed $\mathbf{u}$, by applying a horizontal force, as shown in the figure below. This figure is the same as the case of the rod on rails with the forces relabeled. The normal force replaces the magnetic force, and gravity replaces the resistive force. The vertical component of the normal force does positive work, but the horizontal component does an equal and opposite amount of negative work. You are the entity pumping energy into the system (which shows up as gravitational potential energy), just as you were the entity pumping energy into the above circuit (which showed up as heat). Although the vertical component of the normal force is the only force actually lifting the block upward, the entire normal force does zero net work. Conversely, you are not lifting the block upward, but you do in fact do positive work.



□

Problem 5. A long straight stationary wire is parallel to the y axis and passes through the point $z = h$ on the z axis. A current I flows in this wire, returning by a remote conductor whose field we may neglect. Lying in the xy plane is a square loop with two of its sides, of length b, parallel to the long wire. This loop slides with constant speed v in the $\hat{\mathbf{x}}$ direction. Find the magnitude of the electromotive force induced in the loop at the moment when the center of the loop crosses the y axis. does an equal and opposite amount of negative work. So it would hardly be accurate to say that the magnetic force does work.

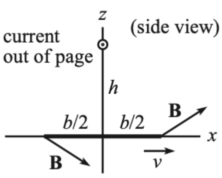
Solution: In the figure below the y axis points into the page. We've arbitrarily chosen the current in the wire to flow in the negative y direction (out of the page), but the sign doesn't matter since all we care about is the magnitude of the emf. At the leading edge of the square loop, the magnitude of B is $\mu_0 I/2\pi r$, where $r = \sqrt{h^2 + (b/2)^2}$. Only the z component matters in the flux, and this brings in a factor of $(b/2)/r$. So

$$B_z = \frac{\mu_0 I}{2\pi r}\frac{b/2}{r} = \frac{\mu_0 I b}{4\pi\left(h^2 + b^2/4\right)}.$$

At the trailing edge, B_z has the opposite sign. If the loop moves a small distance vdt, there is additional positive flux through a thin rectangle with area $b(vdt)$ at the leading edge, and also less negative flux through a similar rectangle at the trailing edge. Both of these effects cause the upward flux to increase. Therefore,

$$\mathcal{E} = \frac{d\Phi}{dt} = 2\frac{b(vdt)B_z}{dt} = 2bvB_z = \frac{\mu_0 I b^2 v}{2\pi\left(h^2 + b^2/4\right)}.$$

The flux is increasing upward. So for our choice of direction of the current in the wire, the induced emf is clockwise when viewed from above, because that creates a downward field inside the loop which opposes the change in flux. For $h = 0$ (or in general for $h \ll b$)$\mathcal{E}$ reduces to $2\mu_0 Iv/\pi$. This is independent of b because the field at the leading and trailing edges decreases with b, while the length of the thin rectangles at these edges increases with b. You can show that our result for $\mathcal{E}$ has the correct units, either by working them out explicitly, or by noting that $\mathcal{E}$ has the units of B (which are the same as $\mu_0 I/2\pi r$) times length squared divided by time, which correctly gives flux per time.



□