Exercise sheet #10

Problem 1. A solenoid has radius R, current I, and n turns per unit length. Given that the magnetic field is $B = \mu_0 n I$ inside and $B = 0$ outside, find the vector potential $\mathbf{A}$ both inside and outside. Hint: Use the expression for the curl in cylindrical coordinates to find the forms of $\mathbf{A}$ that yield the correct values of $\mathbf{B} = \nabla \times \mathbf{A}$ in the two regions.

Solution: First, note that $\mathbf{A}$ must have only a $\hat{\boldsymbol{\theta}}$ component, that is, it must point in the tangential direction around the axis of the solenoid. This follows from the fact that each $d\mathbf{A}$ contribution points in the same direction as the $\mathbf{J}$ current that produces it. And every piece of current in the system points in the $\hat{\theta}$ direction. Furthermore, A_θ can have no dependence on θ or z, by symmetry. So the one nonzero component, A_θ, must be a function only of r. Our goal is therefore to find the function $A_\theta(r)$.

Since we have only one component, $A_\theta(r)$, the only nonzero term in the expression for the curl in cylindrical components is $\hat{\mathbf{z}}(1/r)\partial\left(rA_\theta\right)/\partial r$. So inside the solenoid, $\mathbf{B} = \nabla \times \mathbf{A}$ becomes

$$\hat{\mathbf{z}}\mu_0 n I = \hat{\mathbf{z}}\frac{1}{r}\frac{\partial\left(rA_\theta\right)}{\partial r} \Longrightarrow \frac{\partial\left(rA_\theta\right)}{\partial r} = \mu_0 n I r$$
$$\Longrightarrow rA_\theta = \frac{\mu_0 n I r^2}{2} \Longrightarrow A_\theta = \frac{\mu_0 n I r}{2} \quad \text{(inside)},$$

Outside the solenoid, $\mathbf{B} = \nabla \times \mathbf{A}$ becomes

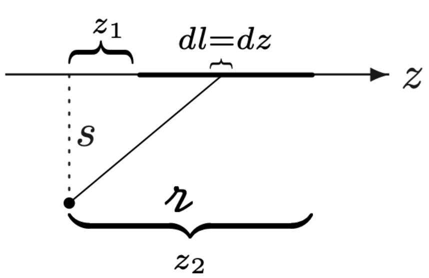
$$0 = \hat{\mathbf{z}}\frac{1}{r}\frac{\partial\left(rA_\theta\right)}{\partial r} \Longrightarrow \frac{\partial\left(rA_\theta\right)}{\partial r} = 0$$
$$\Longrightarrow rA_\theta = C \Longrightarrow A_\theta = \frac{C}{r} \quad \text{(outside)}.$$

We see that any field proportional to $1/r$ yields zero curl outside the solenoid. All fields that are proportional to $1/r$ yield the same zero curl, but not the same line integral around a circle of radius r. This is due to the fact that the curl of $1/r$ diverges at the origin; you should think about how Stokes' theorem comes into play.

Inside the solenoid, if we had included a constant of integration in the expression for the vector potential, we would have obtained an additional term of the form C/r. Although this wouldn't affect the $\mathbf{B} = \mu_0 n I \mathbf{z}$ result at points away from the origin, it would yield an infinite B at $r = 0$. So we must reject this term. □

Problem 2. Find the magnetic vector potential of a finite segment of straight wire carrying a current I. [Put the wire on the z axis, from z_1 to z_2]. Calculate the magnetic field generated by that potential.

Solution: Considering the coordinate system suggested.



$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{I\hat{\mathbf{z}}}{r}dz = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$$
$$= \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Bigg|_{z_1}^{z_2} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}$$

$$\begin{aligned}\mathbf{B} =& \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}\\
=& -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-\left[(z_2)^2+s^2\right]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{z_1^2-\left[(z_1)^2+s^2\right]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}\\
=& -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}},\\
&\text{or, since } \sin\theta_1 = \frac{z_1}{\sqrt{(z_1)^2+s^2}} \text{ and } \sin\theta_2 = \frac{z_2}{\sqrt{(z_2)^2+s^2}},\\
=& \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\boldsymbol{\phi}}.\end{aligned}$$

□

Problem 3. Magnetic scalar potential

(a) Consider an infinite straight wire carrying current I. We know that the magnetic field outside the wire is $\mathbf{B} = (\mu_0 I/2\pi r)\,\hat{\boldsymbol{\theta}}$. There are no currents outside the wire, so $\nabla\times\mathbf{B} = 0$; verify this by explicitly calculating the curl.

(b) Since $\nabla\times\mathbf{B} = 0$, we should be able to write $\mathbf{B}$ as the gradient of a function, $\mathbf{B} = \nabla\psi$. Find ψ, but then explain why the usefulness of ψ as a potential function is limited.

Proof. (a) The curl in cylindrical coordinates is given in Appendix F. Since B has only a $\hat{\boldsymbol{\theta}}$ component, and since this component has only r dependence, the only term in the curl that has a chance of being nonzero is $(1/r)\left(\partial\left(rB_\theta\right)/\partial r\right)\hat{\mathbf{z}}$. But $B_\theta \propto 1/r$, so this term is zero, as desired.

(b) In cylindrical coordinates, the $\hat{\theta}$ component of the gradient of a function ψ is given in Appendix F as $(1/r)(\partial\psi/\partial\theta)\hat{\boldsymbol{\theta}}$. So we want

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\mu_0 I}{2\pi r} \Longrightarrow \frac{\partial\psi}{\partial\theta} = \frac{\mu_0 I}{2\pi} \Longrightarrow \psi = \frac{\mu_0 I}{2\pi}\theta.$$

We therefore see that $\mathbf{B}$ can be written as $\nabla\psi$. However, the problem with this ψ is that it is multi-valued. For example, for given r and z, the values $\theta = 0, 2\pi, 4\pi$, etc., all correspond to the same point in space. So ψ cannot be used as a potential that uniquely labels each point in space. In a limited region, however, it can be of use.

□

Problem 4. Consider an infinite solenoid with circular cross section. The current is I, and there are n turns per unit length. Show that the magnetic field is zero outside and $B = \mu_0 n I$ (in the longitudinal direction) everywhere inside. Do this in three steps as follows.

(a) Show that the field has only a longitudinal component. Hint: Consider the contributions to the field from rings that are symmetrically located with respect to a given point.

(b) Use Ampère's law to show that the field has a uniform value outside and a uniform value inside, and that these two values differ by $\mu_0 n I$.

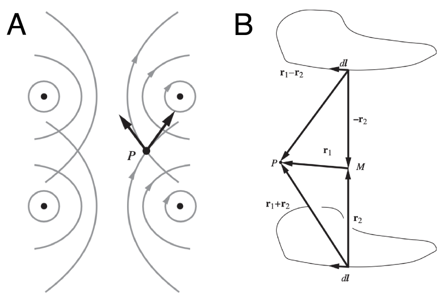
(c) Show that $B \to 0$ as $r \to \infty$. There are various ways to do this. One is to obtain an upper bound on the field contribution due to a given ring by unwrapping the ring into a straight wire segment, and then finding the field due to this straight segment.

Solution: (a) First solution: The longitudinal nature of the field follows from considering the contributions from two loops on either side of a given point P, equidistant from P. You can find the field due to one ring in the notes for lecture 14 page 5, so by superposition the fields due to two rings are as shown in the figure below (A). At any point P on the plane midway between the rings, the magnetic field points in the longitudinal direction, because the radial components cancel, as shown. This argument holds both inside and outside the solenoid, although we will find in part (c) that the field outside is actually zero. You should convince yourself why there are cancelation effects that make it possible for the field to be zero outside, but not inside.

Second solution: We can show that the Biot-Savart contributions from corresponding small intervals of two symmetrically located circles sum to a longitudinal vector. This argument (along with all the other results in this problem) actually holds for a solenoid with a cross-section of arbitrary uniform shape. To see why, consider the Biot-Savart $\boldsymbol{dl} \times \mathbf{r}$ cross products involved in calculating the field at point P in the figure below (B) due to the two dl pieces shown. The point M is midway between the two pieces, and the various vectors are labeled as shown. The sum of the two $\boldsymbol{dl} \times \mathbf{r}$ contributions is

$$dl \times (\mathbf{r}_1 + \mathbf{r}_2) + \boldsymbol{dl} \times (\mathbf{r}_1 - \mathbf{r}_2) = 2dl \times \mathbf{r}_1$$

Now, $\mathbf{r}_1$ points in a transverse direction (that is, perpendicular to the axis) because both M and P lie on the plane midway between the loops. And dl points in a transverse direction too, of course. Since the cross product of two vectors is perpendicular to both vectors, we see that $dl \times \mathbf{r}_1$ points in the longitudinal direction, as desired. This reasoning holds both inside and outside the solenoid.



(b) Having shown that the field is longitudinal, we will now show that it is uniform inside (and outside) the solenoid. It is certainly uniform in the longitudinal direction, by symmetry. So the task is to show that it is uniform in the transverse direction. Consider a rectangular Amperian loop lying completely inside the solenoid, with two sides pointing in the longitudinal direction, and two sides pointing in a transverse direction, as shown in below (A). This loop encloses zero current, so the line integral of $\mathbf{B}$ must be zero. The line integral is nonzero only along the longitudinal sides (this would effectively be true even if there existed a component of $\mathbf{B}$ in the transverse direction, because the contributions would cancel on the two transverse sides of the rectangle). So the field must have the same value on the longitudinal sides. Since the rectangle can have an arbitrary width and be positioned at an arbitrary location inside the solenoid, we conclude that the field must be uniform inside the solenoid. The same reasoning holds outside the solenoid, so the field is uniform there too (with a different value, as we will see).

We must now determine how these two uniform values (inside and outside) are related. If we draw a rectangular Amperian loop with one longitudinal side inside the solenoid and the other outside (let these sides have length ℓ), as shown below (B), the loop now encloses a current of $nI\ell$. Ampère's law therefore gives (taking positive current to point into the page)

$$B_{\text{in}}\, \ell - B_{\text{out}}\, \ell = \mu_0 n I \ell \Longrightarrow B_{\text{in}} = B_{\text{out}} + \mu_0 n I$$

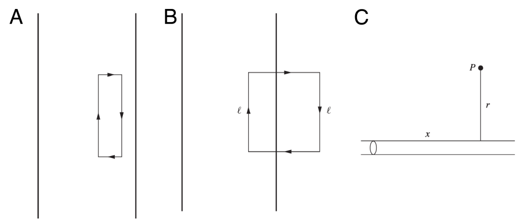
Note that since the transverse width of the rectangle can have any size, this argument by itself shows that the field is uniform inside and outside. We technically didn't need to include the previous paragraph.

(c) The $B_{\text{in}} = B_{\text{out}} + \mu_0 nI$ result, combined with the above results about uniformity, implies that if we can show that the field is zero at any point outside the solenoid, then we have shown that $B = 0$ everywhere outside, and $B = \mu_0 nI$ everywhere inside. And indeed, we can show that $B = 0$ at infinity, as follows.

Consider the field at a point P due to a particular loop of the solenoid. Let P be a large distance from the loop (large compared with the size of the loop). The field due to the loop is smaller than the field due to a straight wire segment that has the same length $b = 2\pi a$ as the loop (where a is the radius of the solenoid) and that is oriented perpendicular to the vector to P. (This is true because the current moves in different directions around the loop, so the Biot-Savart contributions partially, or actually mostly, cancel.) Therefore, an upper bound on the field due to the solenoid is the field due to an infinite set of straight wire segments with length b, lined up side by side. Only the longitudinal components of the fields due to these wires will survive, but we don't need to use this fact. It turns out that we can obtain a sufficiently small upper bound on the field by adding up the magnitudes of the fields due to the wire segments. This certainly overestimates the net field. The magnitude of the field due to a distant wire segment is $(\mu_0/4\pi)\, Ib/\left(x^2 + r^2\right)$, where r is the perpendicular distance from P to the solenoid, and x is the distance shown in the figure below (C). There are n wire segments per unit length, so an upper bound on the field due to the solenoid is

$$B_{\text{bound}} = \frac{\mu_0 nIb}{4\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2 + r^2}$$

The integral here equals $(1/r)\tan^{-1}(x/r)\big|_{-\infty}^{\infty} = \pi/r$. So in the $r \to \infty$ limit, our upper bound on B is zero; B must therefore be zero at $r = \infty$, as we wanted to show. Due to the overestimates we made above, the field actually goes to zero much faster than $1/r$, but our coarse estimates were good enough to get the job done. Another method is to use the fact that the field due to a ring behaves like $1/d^3$ at large distances d.



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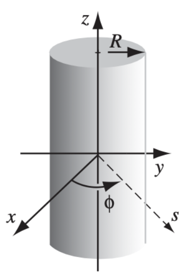
Problem 5. A long circular cylinder of radius R carries a magnetization $\mathbf{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector. Find the magnetic field due to $\mathbf{M}$, for points inside and outside the cylinder.



Solution:

$$\nabla \times \mathbf{M} = \mathbf{J}_b = \frac{1}{s}\frac{\partial}{\partial s}\left(sks^2\right)\hat{\mathbf{z}} = \frac{1}{s}\left(3ks^2\right)\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = ks^2(\hat{\phi} \times \hat{\mathbf{s}}) = -kR^2\hat{\mathbf{z}}.$$

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the total current should be zero ... is it? Yes, for $\int J_b da = \int_0^R (3ks)(2\pi s ds) = 2\pi k R^3$, while $\int K_b dl = \left(-kR^2\right)(2\pi R) = -2\pi k R^3$.] Since these currents have cylindrical symmetry, we can get the field by Ampère's law:

$$B \cdot 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 \int_0^s J_b da = 2\pi k \mu_0 s^3 \Rightarrow \mathbf{B} = \mu_0 k s^2 \hat{\phi} = \mu_0 \mathbf{M}.$$

Outside the cylinder $I_{\text{enc}} = 0$, so $\mathbf{B} = 0$. □