Lecture 17. Magnetic fields in matter II. The auxiliary field H

Lost time:

* Types of magnetic materials
  - diamagnets
    - magnetization opposite to external $\underline{B}$
    - repelled "feebly" by external $\underline{B}$
    - diamagnetism originates from "tilt" of e^- orbits
  - paramagnets
    - magnetization // to external $\underline{B}$
    - attracted by external $\underline{B}$
    - paramagnetism originates from the spin of e^-
  - ferromagnets
    - Retain magnetization in absence of external $\underline{B}$
    - Magnetization depends on "history"

* Magnetic dipole $\longrightarrow$ $\underline{m} = I\underline{a}$; Force on a dipole $\underline{F} = \nabla(\underline{m}\cdot\underline{B})$; torque on a dipole $\underline{\tau} = \underline{m}\times\underline{B}$

* Magnetization $\rightarrow$ $M \equiv \dfrac{\text{magnetic dipole moment}}{\text{unit volume}}$; $\underline{M} = \dfrac{1}{V}\sum_i \underline{m}_i$

* Bound currents
  - $\underline{K}_b = \underline{M}\times\hat{n}$ bound surface current.
  - $\underline{J}_b = \nabla\times\underline{M}$ bound volume current

Today: Ampère's law in magnetized materials

The effect of magnetization is to establish the bound currents $\underline{J}_b$ and $\underline{K}_b$. Similar to the case for dielectrics, the field of a magnetized material is the field due to bound currents + the field due to everything else, which we will call free current. Then the total current will be:

$$\underline{J} = \underline{J}_b + \underline{J}_f$$

total current is equal to bound + free current.

$\underline{J}_b$: due to magnetization

$\underline{J}_f$: actual transport of charge (e.g. due to a battery)

In terms of this total current we can write Ampère's law as:

$$\nabla\times\underline{B} = \mu_0\underline{J} = \mu_0(\underline{J}_b + \underline{J}_f)$$

$$\Rightarrow \frac{1}{\mu_0}\nabla\times\underline{B} = \Big[\underbrace{\nabla\times\underline{M}}_{\text{this is the definition of } \underline{J}_b} + \underline{J}_f\Big]$$

$$\Rightarrow \nabla\times\underbrace{\left(\frac{1}{\mu_0}\underline{B} - \underline{M}\right)}_{\equiv\, \underline{H} \text{ auxiliary field}} = \underline{J}_f$$

with this definition we have:

$$\boxed{\nabla\times\underline{H} = \underline{J}_f \qquad \oint \underline{H}\cdot d\underline{l} = I_{f\,enc}}$$

or in integral form:

with $\underline{H} \equiv \dfrac{1}{\mu_0}\underline{B} - \underline{M}$

where $I_{f\,enc}$ is the total free current passing through the amperian loop.

Here $\underline{H}$ plays the role that $\underline{D}$ played for dielectrics. So if symmetry allows it we can calculate $\underline{H}$ directly by the usual ampere method.

~~As for the electrodynamics dielectric case~~

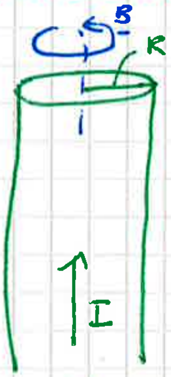
From the expressions above it'd seem that $\underline{H}$ is the same as $\underline{B}$, but there are significant differences.

We know that the divergence of $\underline{B}$ is zero, but in general the divergence of $\underline{H}$ is NOT zero:

$$\nabla \cdot \underline{H} = \nabla \cdot \left(\frac{1}{\mu_0} \underline{B} - \underline{M} \right) = -\nabla \cdot \underline{M}$$

Not only free currents, but also the magnetization is a source of $\underline{H}$. This is trivial if we think about it physically, but might not be obvious looking at the equations. Think for example of a bar magnet, in this case there are no free currents, but there is a magnetic field. So not even if I_{free} is zero, this doesn't imply that $\underline{H}$ is zero.

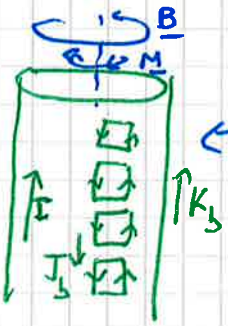
EXAMPLE: A long copper rod of radius R carries a uniformly distributed (free) current I. Find $\underline{H}$ inside and outside the rod.



We have a wire of radius R, that carries a current I in the direction of the axis of the wire.

I is longitudinal
$\Rightarrow \underline{B}$, will be circumferential by the right hand rule.

In which direction will $\underline{M}$ point?
Copper is weakly diamagnetic $\rightarrow$ dipoles will line up opposite to the field creating a bound current antiparallel to I

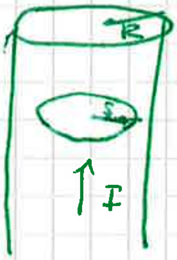


We don't know yet the magnitude of $\underline{J}_b$ or $\underline{K}_b$ but we can say that they will generate a circumferential field $\underline{M}$

Now we will use the eq. for Ampère's law to calculate $\underline{H}$:

$$\oint \underline{H} \cdot d\underline{l} = I_{fenc}$$

Let's draw an Amperian loop:



For a loop of radius $s < R$

$$\oint \underline{H} \cdot d\underline{l} = \underline{H} \int d\underline{l} = \underline{H}(2\pi s) = \underbrace{I_{fenc}}_{\text{what is } I_{fenc}?} \quad (*)$$

If the current is uniformly distributed:

$$\underline{J} = \frac{I}{\pi R^2} \quad \text{definition of } \underline{J} \text{ (current/unit area)}$$

In the amperian loop the current enclosed is proportional to the area of the loop:

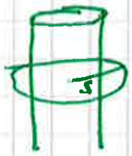
$$I_{fenc} = J \cdot \pi s^2 = \frac{I \, \pi s^2}{\pi R^2} = \frac{I s^2}{R^2}$$

So subs this in eq (*) we get:

$$\underline{H}(2\pi s) = \frac{I s^2}{R^2}$$

$$\Rightarrow \boxed{\underline{H} = \frac{I}{2\pi R^2} s \hat{\underline{\Phi}} \quad (\text{for } s < R) \text{ inside the wire}}$$

Outside of the wire the current enclosed by the amperial loop will be the total free current I:



$$H(2\pi s) = I_{enc} = I$$

$$\Rightarrow \boxed{\underline{H} = \frac{I}{2\pi s}\hat{\Phi} \quad \text{for } s \geq R \text{, outside of the wire.}}$$

In the outside region (as always in empty space) $\underline{M} = 0$ so

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} = \frac{1}{\mu_0}\underline{B} = \frac{I}{2\pi s} \Rightarrow \boxed{\underline{B} = \frac{\mu_0 I}{2\pi s}\hat{\Phi}}$$

the same as a non-magnetized wire.

Inside the wire we cannot determine $\underline{B}$, since we don't know $\underline{M}$.

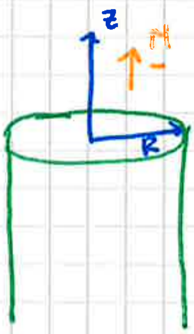
EXAMPLE: An infinitely long cylinder of radius R, carries a frozen in magnetization $\underline{M}$, parallel to the axis

$$\underline{M} = ks\hat{z} \qquad \text{where } k \text{ is a constant}$$
s is the distance from the axis

If there is no free current anywhere, find the magnetic field inside & outside the cylinder.

→ We will solve this problem by 2 methods.
1) We will calculate $\underline{J}_b$ and $\underline{K}_b$ and from there $\underline{B}$
2) We will use Ampère's law for $\underline{H}$ and then calculate $\underline{B}$.

As always we first make a sketch to understand the problem & its symmetries:



The formulas we need to calculate the bound currents are:

$$\underline{K}_b = \underline{M} \times \hat{n}$$

$$\underline{J}_b = \nabla \times \underline{M}$$

We have a cylinder so it's best to use cylindrical coordinates. In this case, for any vector $\underline{A}$:

$$\nabla \times \underline{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta}$$
$$+ \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{z} \qquad \text{(Appendix F purcell p. 792)}$$

First let's calculate $\underline{J}_b$:
In this case $M \parallel z$ so the only terms that survive will be:

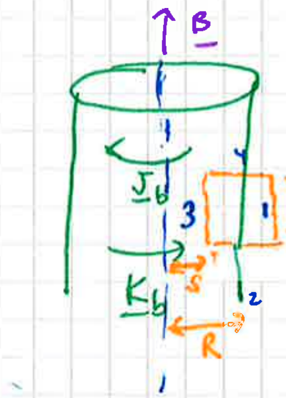
$$\underline{J}_b = \nabla \times \underline{M} = \frac{1}{s}\left[\frac{\partial}{\partial \theta}(ks)\right]\hat{r} - \frac{\partial}{\partial s}(ks)\hat{\theta} = -k\hat{\theta}$$

And now to calculate $\underline{K}_b$ we notice that in this case $\hat{n}$ is the radial direction. At the surface of the cylinder $s = R$, so:

$$\underline{K}_b = \underline{M} \times \hat{n} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & kR \\ 1 & 0 & 0 \end{vmatrix} = kR\hat{\theta}$$

$$\therefore \underline{J}_b = -k\hat{\theta} \quad \text{and} \quad \underline{K}_b = kR\hat{\theta}$$

We see that these bound currents are similar to two solenoids:



$\underline{B}$ is in the $\hat{z}$ direction, basically $\underline{J}_b$ & $\underline{K}_b$ are equivalent to 2 solenoids. Due to symmetry & the absence of free currents $\boxed{\text{outside } \underline{B} = 0}$.

Now for the region inside the cylinder we consider the loop drawn

$$\oint \underline{B} \cdot d\underline{l} = \underbrace{\int_1 \underline{B} \cdot d\underline{l}}_{=0} + \underbrace{\int_2 \underline{B} \cdot d\underline{l}}_{=0} + \underbrace{\int_3 \underline{B} \cdot d\underline{l}}_{=Bl} + \underbrace{\int_4 \underline{B} \cdot d\underline{l}}_{=0} = \mu_0 I_{enc} \quad (*)$$

Sides 2 & 4 = 0 because they are $\perp$ to $\underline{B}$. In region 1 $\underline{B} \cdot d\underline{l} = 0$ because outside the solenoid $\underline{B} = 0$

From eq (*) we have that:

$$B = \frac{\mu_0 I_{enc}}{l} \quad (\odot)$$

What is I_{enc}? it's the sum of the bound currents:

Remembering that by definition

$$I_J = \int \underline{J} \cdot d\underline{a} \quad \text{and} \quad I_K = \int \underline{K} \cdot d\underline{l} \quad \text{and} \quad I_{enc} = I_J + I_K$$

$$I_{enc} = \int \underline{J}_b \cdot d\underline{a} + \int \underline{K}_b \cdot d\underline{l} = \int -K\,da + \int KR\,dl = -K\int da + KR\int dl$$

subs the expressions we found for $\underline{J}_b$ & $\underline{K}_b$

$$= -K[l(R-s)] + KRl = -KRl + Kls + KRl = Kls$$

So subs I_{enc} in eq $(\odot)$:

$$B = \frac{\mu_0}{l}(Kls) = \mu_0 Ks \quad \therefore \boxed{\underline{B} = \mu_0 Ks\,\hat{z} \quad \text{inside}}$$

2) Now we will use Ampère's law for $\underline{H}$ to then calculate $\underline{B}$.

$$\oint \underline{H} \cdot d\underline{l} = I_{fenc}$$

To calculate $\underline{H}$ we use the same amperian loop, because there are no free currents $\underline{H} = 0$

$$\oint \underline{H} \cdot d\underline{l} = Hl = \mu_0 I_{fenc} = 0 \quad \therefore \boxed{\underline{H} = 0}$$

Now $\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} = 0 \quad (\Delta)$

Outside the cylinder $\underline{M} = 0 \Rightarrow \boxed{\underline{B} = 0 \text{ outside}}$

Inside the cylinder $\underline{M} = Ks\,\hat{z}$. So subs this in eq (Δ):

$$\frac{1}{\mu_0}\underline{B} = Ks\,\hat{z} \Rightarrow \boxed{\underline{B} = \mu_0 Ks\,\hat{z} \text{ inside}}$$

Linear and non-linear media

For paramagnetic & diamagnetic materials, magnetization is sustained in the presence of an external $\underline{B}$, after it's turned off, it disappears.

In most materials $\underline{M} \propto \underline{B}$ magnetization is proportional to the field.

In the case of the magnetic field, the proportionality is given in terms of $\underline{H}$: (by convention)

$$\boxed{\underline{M} = \chi_m \underline{H}} \qquad \left(\text{remember } \underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M}\right)$$

* where χ_m is the magnetic susceptibility.
* From the last eq you can see it's a dimensionless qty.
* it is + for paramagnets
  ↳ − for diamagnets
* Materials that obey this relation are called linear media.

We can re-write the last equation in terms of $\underline{B}$, subs the definition of $\underline{H}$:

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} \Rightarrow \underline{B} = \mu_0(\underline{H}+\underline{M}) \underset{\substack{\uparrow \\ \text{subst the last eq for linear media}}}{=} \mu_0(\underline{H} + \chi_m \underline{H}) = \underbrace{\mu_0(1+\chi_m)}_{\equiv \mu}\underline{H}$$

$\underline{M}$ from

$$\therefore \boxed{\begin{array}{l} \underline{B} = \mu \underline{H} \\ \text{where } \mu = \mu_0(1+\chi_m) \end{array}} \leftarrow \text{permeability of the material}$$

In vacuum, where there is no material to magnetize $\chi_m = 0$ so $\mu = \mu_0$ permeability of free space.

Even in the case of linear media we still have that $\nabla \cdot \underline{H} \neq 0$ so $\underline{B}$ and $\underline{H}$ are NOT equivalent:

$$\nabla \cdot \underline{H} = \nabla \cdot \left(\frac{1}{\mu}\underline{B}\right) = \frac{1}{\mu}\nabla \cdot \underline{B} + \underline{B} \cdot \nabla\left(\frac{1}{\mu}\right) = \underline{B} \cdot \nabla\left(\frac{1}{\mu}\right)$$

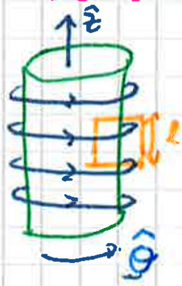
so $\underline{H}$ is NOT divergenceless at points where μ is changing.

For example at the boundary between a linear material and outside μ will change.

In terms of current, for linear materials we do have that the bound current is proportional to the free current:

$$\underline{J}_b \underset{\substack{\uparrow \\ \text{by definition}}}{=} \nabla \times \underline{M} \underset{\substack{\uparrow \\ \text{for linear media}}}{=} \nabla \times (\chi_m \underline{H}) \underset{\substack{\uparrow \\ \text{because } \nabla \times \underline{H} = \underline{J}_f \text{ by definition}}}{=} \chi_m \underline{J}_f$$

EXAMPLE: An infinite solenoid (n turns/unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.



We can't compute $\underline{B}$ directly since it's due partially to bound currents. But we can use Ampère's law for $\underline{H}$:

$$\oint \underline{H} \cdot d\underline{l} = I_{fenc} \quad \Rightarrow \quad H \oint d\underline{l} = I_{fenc}$$

so as usual this reduces to calculating $\oint d\underline{l}$ along a loop and I_{enc}:

Using the same arguments as the previous example, we see that along the loop drawn:

$$\oint d\underline{l} = l$$

On the other hand, the total free current enclosed is $I_{fenc} = NI$ where N is the total number of turns. So with this our equation yields:

$$Hl = NI \Rightarrow \underline{H} = \frac{NI}{l}\hat{z} = nI\hat{z}$$

by definition $n = \frac{N}{l}$ # of turns / unit length

Now using that $\underline{B}$ is proportional to $\underline{H}$ via the permeability of the material, i.e $\underline{B} = \mu \underline{H}$

$$\Rightarrow \underline{B} = \mu \underline{H} = \mu_0 (1 + \chi_m) nI\hat{z}$$

because $\mu \equiv \mu_0(1+\chi_m)$

If the material is paramagnetic the field is slightly enhanced. If it's diamagnetic the field is slightly reduced. We can see this since the surface bound current is in the same direction as I:

$$\underline{K}_b = \underline{M} \times \hat{n} = \chi_m (\underline{H} \times \hat{n}) = \chi_m n I \hat{\theta}$$

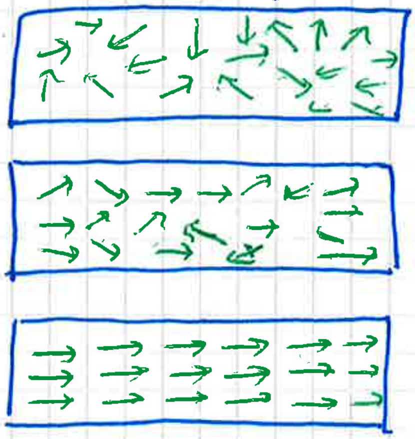
Ferromagnets

Like paramagnetism, ferromagnetism involves the alignment of magnetic dipoles associated with the spins of unpaired electrons.

The feature that distinguishes paramagnets from ferromagnets is that nearby dipoles tend to align w/ each other, i.e each dipole likes to point in the same direction as its neighbours, creating "small" "patches" of alignment known as domains.

Each domain contains billions of dipoles all lined up, which are actually visible using a microscope. However, the domains are randomly oriented w/ respect to each other, so the material as a whole is not magnetized.

We can think of the following pictures for an iron bar:



unmagnetized iron

slightly magnetized iron

strongly magnetized iron

We can produce this transition in iron by applying an external magnetic field or lowering the temperature in the absence of an external $\underline{B}$.

each arrow represents a magnetic domain i.e a cluster of atoms w/ their spins aligned.

Why do dipoles aligned in ferromagnetic materials?

① Quantum effect: It is energetically favorable for the spins of adjacent atoms to be parallel.
Pauli exclusion principle + Coulomb interaction between electrons: When 2 electrons are closed to each other, their wave functions overlap → spins need to align to minimize energy of system.

② Long-range effect: As more domains align, ~~they will~~ the magnetization will produce a field that aligns domains that are farther away.

③ Reducing temperature: Random thermal motions compete with alignment of domains.
For certain materials, such as iron, at a very well defined temperature, there is a spontaneous "lining up" of ~~mass~~ atomic magnetic moments. i.e.
↑
meaning w/NO external $\underline{B}$

For iron, the temp. at which it becomes ferromagnetic is below 770 C, this transition temperature is known as the curie point.

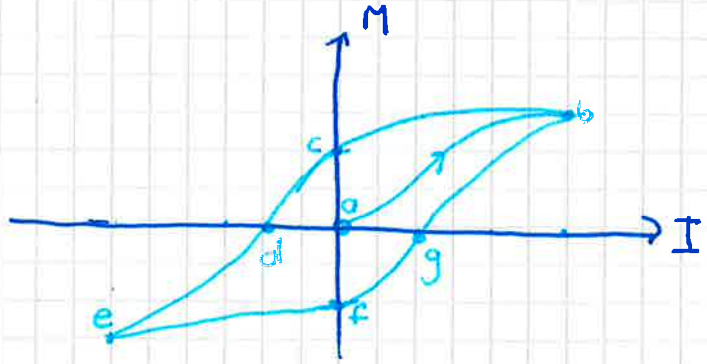
The curie point is like a boiling or freezing point in the sense that the behavior of the material changes abruptly. Above 770C iron is a paramagnet, below a ferromagnet. It is not a gradual, but a sharp change → phase transition.

Demo 548. Electromagnet.

A key feature of ferromagnetic materials is that once the externally applied $\underline{B}$ is switched off, the alignment of moments induced stays.

When the field is switched off there will be some return to the original orientation of the domains, but there will remain aligned domains → permanent magnet.

Let's think about what happens as we run a current through a piece of iron as in the demo:



a → b we turn on the current until we align all the dipoles in the material. This is called the saturation point. We induced a maximum $\underline{M}$

b → c as we reduce the current, we DON'T return to point a. Rather there is some residual alignment, so we have a non-zero $\underline{M}$. we have a permanent magnet

c → e to remove the magnetization, we need to run a current in the opposite direction (I negative). We cross the point d where M = 0 and as we increase neg I we arrive at point e. All dipoles are aligned in the opposite direction.

e → f if now I switch off the current, some but not all of the dipoles will return to the original state, so now again I have a permanent nonzero magnetization

f → b To "complete the story" I can turn on again I in the positive direction, achieve M = 0 at point g and if I continue increasing I, I will arrive again at the the point of saturation b.

this path is known as a hysteresis loop.
from the Greek "lag behind"

We can now see what we meant when we said that for ferromagnets, the magnetization depends on the material's history.

This is also how an electromagnet works. We apply current to a ferromagnetic material & we induce a strong permanent $\underline{M}$.
Wrap a coil around a piece of iron and add a current source (e.g. battery) and you have your own electromagnet. Fields of iron magnets are usually $\sim$ 0.5 tesla

Demo. 542 hysteresis loop.