Lecture 16. Magnetic fields in matter I. Mechanisms of magnetization. Bound currents.

Last time:

* Magnetic vector potential

$\underline{B} = \nabla \times \underline{A}$ where $\underline{A}$ is the magnetic vector potential

* Poisson's equation for $\underline{A}$:

$\nabla^2 A = -\mu_0 \underline{J}$; $\nabla \cdot \underline{A} = 0$

the solutions to ~~these~~ this equation are:

$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{J(\underline{r}) dV}{r}$ for volume current densities

$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{K(\underline{r})}{r} da$ for surface current density

$A(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{I}(\underline{r})}{r} dl$: for line currents

Remember: r is the distance from the source to the point we are calculating the potential at.

* Magnetic vector potential of a dipole:

$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{r}}{r^2}$; where $\underline{m} \equiv I\underline{a}$ magnetic dipole moment

Today: Diamagnets, paramagnets, ferromagnets

Demo 566, 602. How do various substances respond to the magnetic field?

So far we have seen that currents are sources of magnetic fields.
We have pictured current carrying wires, how do we align this picture w/ our everyday experience of magnetic phenomena? e.g. compass needles, refrigerator magnets...
where are the current carrying wires?
Just as we did for dielectrics we have to think about what happens at the atomic scale. At this scale we find "tiny" currents → electrons orbiting around nuclei & spinning around their axes. Ordinarily these ~~random~~ currents cancel because of the random orientation of atoms. But when a magnetic field is applied, these dipoles align and the material becomes magnetized.

Magnetization differs from electric polarization, in which polarization occurs generally in the direction of the externally applied $\underline{E}$. Different materials will acquire magnetization in different directions with respect to ~~Bo~~ the externally applied $\underline{B}$.

1) Diamagnets: Materials that acquire magnetization opposite to the externally applied $\underline{B}$. Diamagnetism is a property of every atom + molecule. ~~[illegible]~~, when this behavior is not observed, it's because it's outweighted by another effect. (But it doesn't mean it's not there). Diamagnetic ~~[illegible]~~ substances are "feebly" repelled by the externally applied $\underline{B}$. examples are water, salt, diamond, etc.

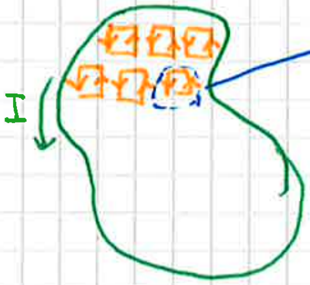
2) Paramagnets: Materials that acquire magnetization parallel to the externally applied $\underline{B}$. These substances are attracted towards the region of stronger field. In some metals paramagnetism is ~ = to diamagnetism, but in some others e.g. $NiSO_4$ $CuCl_2$ the paramagnetic effect is much stronger. Paramagnetism increases as the temperature is lowered.

3) Ferromagnets: For these materials the magnetization is not determined by the presence of an external $\underline{B}$, rather by the whole magnetic "history". They retain magnetization even in the absence of an external $\underline{B}$. Examples of these materials are iron, magnetite.

Magnetization origins
Similar to the argument we made to understand electric polarization, to understand magnetization we need to look at the molecular scale. Magnetic materials consist of many dipoles (permanent or induced) which are all aligned. Thus to understand this we need to understand the source of a dipole, and the force + torque it is subjected to due to an externally applied $\underline{B}$.

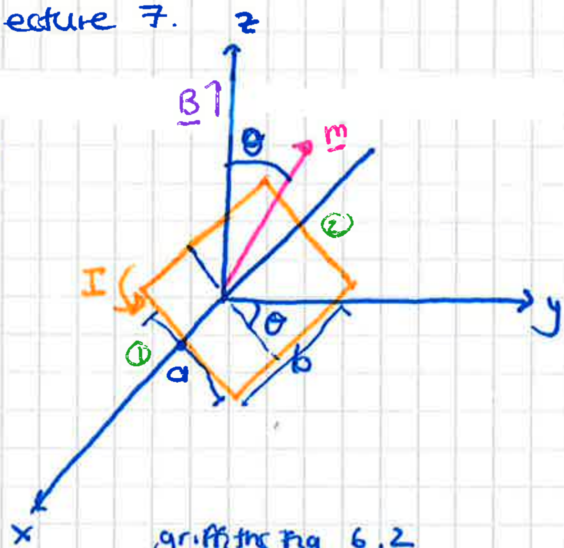
Torque and Force on a magnetic dipole

We will revisit now the calculation at the end of lecture 7.



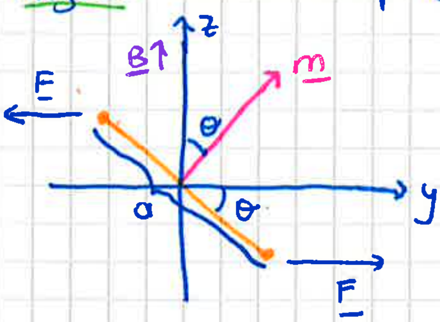
we take one of these loops, center it around the origin and tilt it an angle θ from the z axis towards the y axis.

Any current loop can be built from "infinitesimal" rectangles, which internal sides cancel.



griffiths fig 6.2
loop centred at origin
θ tilt from z axis towards y axis
$\underline{B}$ points in the z direction

As we saw on lecture 7 the net $\underline{F}$ on the loop is zero, but there is a torque on the loop due to the forces on the horizontal sides of the loop. (regions ① + ②) in the drawing).



$\tau = a F \sin\theta\, \hat{x}$ (Remember $\underline{\tau} = \underline{r} \times \underline{F}$)

and the magnitude of the force is:
$F = IbB$ (Remember $\underline{F} = I(\underline{l} \times \underline{B})$)

$\therefore\ \tau = \underbrace{Iab}_{\equiv m} B \sin\theta\, \hat{x}$

$\equiv m$ magnetic dipole moment of the loop.

direction of I + m are related by the right hand rule.
Thumb points in direction of I,
fingers in the direction of m.

$\therefore$ $\boxed{\underline{\tau} = \underline{m} \times \underline{B}}$ torque on any localized current distribution in the presence of a uniform field.

where $\boxed{\underline{m} = I\underline{a}}$ magnetic dipole moment

Notice that this eq. for torque has the same form as the electrical analogue:

$\underline{\tau} = \underline{p} \times \underline{E}$. for $\underline{p}$ an electric dipole

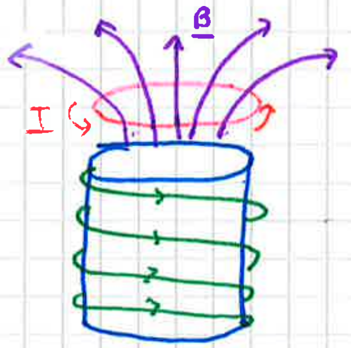
Notably, the torque is in the direction that aligns the dipole parallel to the external field. This is the microscopic origin of paramagnetism. We can consider every electron as a spinning sphere of charge, which constitutes a magnetic dipole.

For a uniform $\underline{B}$ field we saw in lecture 13 that the net force on a current loop is zero:

$$\underline{F} = I \oint (d\underline{l} \times \underline{B}) = I \underbrace{\left(\oint d\underline{l}\right)}_{\text{this is zero over a closed loop.}} \times \underline{B} = 0$$

because $\underline{B}$ is constant we can take it out of the integral.

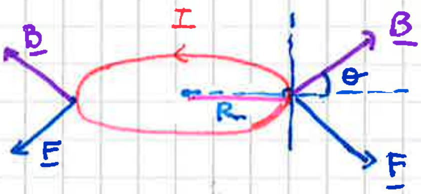
This is NO longer true for a non-uniform field. Let's consider the field generated by a solenoid, we saw in lecture 13 that the field lines look like:



in this case $\underline{B}$ is non-uniform

What happens if we place a current loop suspended above the solenoid?

In this region $\underline{B}$ has a radial component, so there will be a NET downward force on the loop:



Downward force on loop due to the radial component of $\underline{B}$:

$$F = 2\pi I R B \cos\theta$$

In general for an infinitesimal loop with dipole moment $\underline{m}$, in a field $\underline{B}$, the force is:

$$\boxed{\underline{F} = \nabla(\underline{m} \cdot \underline{B})}$$

Origin of diamagnetism. The effect of $\underline{B}$ on atomic orbits

An electron revolving around the nucleus of an atom generates a current. For simplicity we will assume that the electron revolves around the nucleus with a circular orbit of radius R, and the current it generates is steady.

For a circular orbit: $T = \dfrac{2\pi R}{v}$ period of orbit.

so the current generated by the electron is:

$$I = \frac{q}{t} = -\frac{e}{T} = \frac{-ev}{2\pi R}$$

where e is the charge of the electron, the negative sign comes from the sign of the charge.

So the magnetic dipole moment that corresponds to this current is:

$$\underline{m} = I\underline{a} = \left(-\frac{ev}{2\pi R}\right)(\pi R^2)\,\hat{z} = -\frac{1}{2} evR\,\hat{z}$$

$\leftarrow$ direction is given by right hand rule we can re-write this last equation in terms of orbital angular momentum:

Remembering that the angular momentum is a vector with magnitude: $L = m_e V R$ where m_e is the mass of the electron, pointing in the same direction as $\underline{m}$, so:

$$\boxed{\underline{m} = \frac{-e}{2m_e}\underline{L}}$$

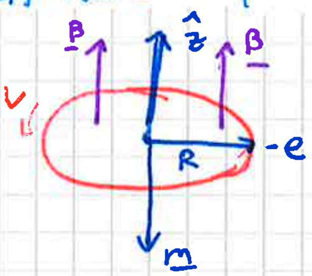
where $\underline{L}$ is the orbital angular momentum.

If $\underline{L}$ is conserved, $\underline{m}$ remains constant in magnitude & direction.

Note: ~~This torque~~ The magnetic dipole moment due to $\bar{e}$ orbiting also produces a torque $\underline{\tau} = \underline{m} \times \underline{B}$ that contributes to paramagnetism. However it's not enough to tilt the orbit, so the contribution of this $\underline{m}$ towards paramagnetism is negligible.

In the absence of an external $\underline{B}$, the electron in orbit has a centripetal acceleration V^2/R sustained by electrical forces alone:

$$\underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}}_{\text{electric force}} = \underbrace{m_e \frac{V_0^2}{R}}_{ma} \quad (*)$$



Now, if we turn on an external magnetic field $\underline{B}$, the electron will experience an additional force $-e(\underline{V} \times \underline{B})$ due to $\underline{B}$. Let $\underline{B}$ be $\perp$ to the orbit:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + eVB = m_e\frac{V^2}{R} \quad (**)$$

to determine the change in magnetic moment we need to determine the change in speed once the external field $\underline{B}$ is on, i.e. $\Delta V = V - V_0$

From eq (**) we have:

$$eVB = m_e\frac{V^2}{R} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} \underset{\text{using eq }(*)}{=} \frac{m_e}{R}(V^2 - V_0^2) \underset{\text{factorizing}}{=} \frac{m_e}{R}(V+V_0)\underbrace{(V-V_0)}_{\Delta V}$$

$$\Rightarrow \Delta V = \frac{e\cancel{V}BR}{m_e\underbrace{(V+V_0)}_{\approx 2V}} = \frac{eBR}{2m_e} \quad (***)$$

$\approx 2V$ assuming the change is small (i.e. $V \sim V_0$)

so with this we can calculate the change in magnetic dipole moment:

$$\Delta\underline{m} = -\frac{1}{2}e(\Delta V)R\hat{z}$$

(Remember we calculated m for an electron in its orbit $\underline{m} = -\frac{1}{2}eVR\hat{z}$)

$$= -\frac{1}{2}e\left(\frac{eBR}{2m_e}\right)R\hat{z}$$

$$= -\frac{e^2R^2}{4m_e}\underline{B}$$

notice that the change in $\underline{m}$ is opposite to $\underline{B}$ i.e. the extra dipole moment is anti parallel to the external field $\rightarrow$ diamagnetism.

We are seeing that the electron "speeds up". Wait! I thought we had said magnetic forces don't work. This is still correct. Up till now we haven't accounted for the fact that a changing magnetic field induces an electric field, which is the one that does the work in this case. (i.e accelerates the electrons). This kind of violates the assumption that this is magnetostatics. In the end it's not worth looking at the details since This is still a qualitative picture, paramagnetism is truly a quantum phenomenon.

Magnetization

The presence of an external field $\underline{B}$ will induce an alignment of dipoles that will give rise to magnetic polarization.

paramagnetism $\Rightarrow$ $\bar{e}$ spins give rise to dipole moments in the direction of $\underline{B}$

diamagnetism $\Rightarrow$ $\bar{e}$ orbiting give rise to dipole moments in the opposite direction of $\underline{B}$

whatever the reason:

$$\boxed{M \equiv \text{magnetization} \equiv \text{magnetic dipole moment / unit volume}} \qquad \underline{M} = \frac{1}{\underset{\uparrow \text{volume}}{V}} \sum_i \underline{m}_i$$

this is analogous to polarization $\underline{P}$ in electrostatics. As we did for electrostatics we will not worry about how $\underline{M}$ got there, just look at the consequences of it.

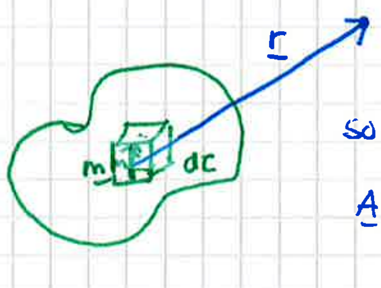
The field of a magnetized object

Let's assume we have a piece of magnetized material, we know the magnetic dipole moment / unit volume. We want to know what is the field produced by the piece of material?

Let's consider first a single magnetic dipole $\underline{m}$. Last lecture we derived that for a dipole the magnetic vector potential is:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{r}}{r^2}$$

Let's consider the following piece of magnetized material:



by definition of magnetization, each volume element $d\tau$ carries a dipole moment $\underline{M}\, d\tau$.

so integrating to obtain the total vector potential:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{M}(\underline{r}) \times \hat{r}}{r^2}\, d\tau$$

as in the electrical case, we will re-write this integral using the identity $\nabla \frac{1}{r} = \frac{\hat{r}}{r^2}$

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \underbrace{\left[\underline{M}(\underline{r}') \times \left(\nabla \frac{1}{r}\right)\right]}_{\text{we can re-write this integral using product rule (v) Griffiths page 21,}} d\tau$$

$$\nabla \times \left(\frac{1}{r} \underline{M}\right) = \frac{1}{r}(\nabla \times \underline{M}) - \underline{M} \times \left(\nabla \frac{1}{r}\right)$$

$$\Rightarrow \underline{M} \times \left(\nabla \frac{1}{r}\right) = -\nabla \times \left(\frac{1}{r} \underline{M}\right) + \frac{1}{r}(\nabla \times \underline{M})$$

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int\left[-\nabla\times\left(\frac{1}{r}\underline{M}\right) + \frac{1}{r}(\nabla\times\underline{M})\right]d\tau$$

And using the divergence theorem we can swap the 1st volume $\int$ for a surface one:

$$\underline{A}(\underline{r}) = \underbrace{\frac{\mu_0}{4\pi}\oint\frac{1}{r}[\underline{M}\times d\underline{a}]}_{\text{looks like the potential of a surface current}} + \underbrace{\frac{\mu_0}{4\pi}\int\frac{1}{r}[\nabla\times\underline{M}]d\tau}_{\text{looks like the potential of a volume current}}$$

we can re-write this as:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\oint_S\frac{\underline{K}_b(\underline{r})da}{r} + \frac{\mu_0}{4\pi}\int_V\frac{\underline{J}_b}{r}d\tau$$

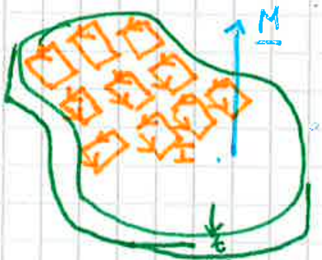
where: $\boxed{\begin{aligned}\underline{K}_b &= \underline{M}\times\hat{n} \quad \text{bound surface current}\\ \underline{J}_b &= \nabla\times\underline{M} \quad \text{bound volume current}\end{aligned}}$

(Analogue of bound volume & surface charge
$\rho_b = -\nabla\cdot\underline{P}$
$\sigma_b = \underline{P}\cdot\hat{n}$)

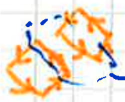
Then, the ~~vector~~ vector potential of a magnetized object is the same as the one produced by a volume current density + surface current density.
As in the case for dielectrics, all of our problems will be solved by finding these bound currents and calculating the fields generated by them.

Physical interpretation of the bound currents
We will make very similar arguments as we did in lecture 9 to convince ourselves that bound currents are real physical currents.

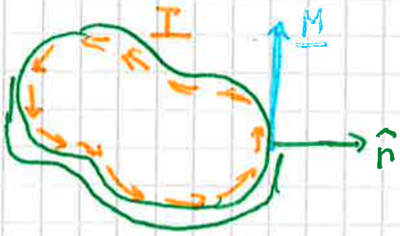
Let's consider a slab of uniformly magnetized material:



If we imagine this material full of tiny loops that each produce a dipole moment:



so we will get some effective surface current that looks like:



What is the current I in terms of M?

Let's assume that each one of the tiny loops has an area a and thickness t.

So in terms of magnetization the dipole moment is:

$$M = \frac{m}{V} \quad \begin{matrix}\leftarrow \text{dipole moment} \\ \leftarrow \text{volume}\end{matrix} \quad \Rightarrow \quad m = M(at) \ *$$

And we also know that in terms of the circulating current the ~~magnetization~~ dipole moment is:

$$m = Ia \ **$$

Then equating equations * and ** we get:

$$Mat = Ia \Rightarrow \underbrace{\frac{I}{t}}_{\equiv K_b} = M \Rightarrow K_b = M$$

current/area, in this case t is the thickness

and we can see in the drawing in the last page that $\underline{M} \times \hat{n}$ is in the direction of K_b.

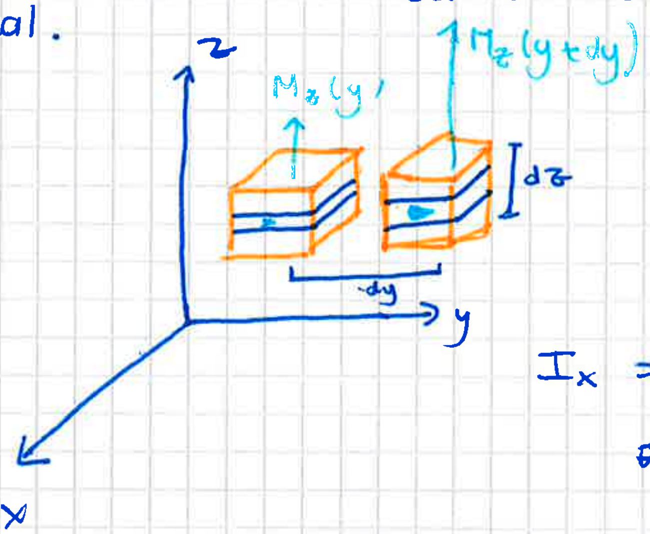
$$\therefore \ \underline{K}_b = \underline{M} \times \hat{n}$$

this expression also implies that there is no current on the top or bottom of the slab because there $\underline{M} \parallel \hat{n}$ so the cross product vanishes.

If we think about it physically, there is no one charge that is moving around the boundary, because the currents are bound.
i.e each charge moves in a tiny loop in an atom, but it produces a magnetic field just like any other current.

Now what happens if the magnetization is non-uniform?

In this case the internal currents of adjacent loops will no longer cancel, so there will be a net current between adjacent chunks of magnetized material.



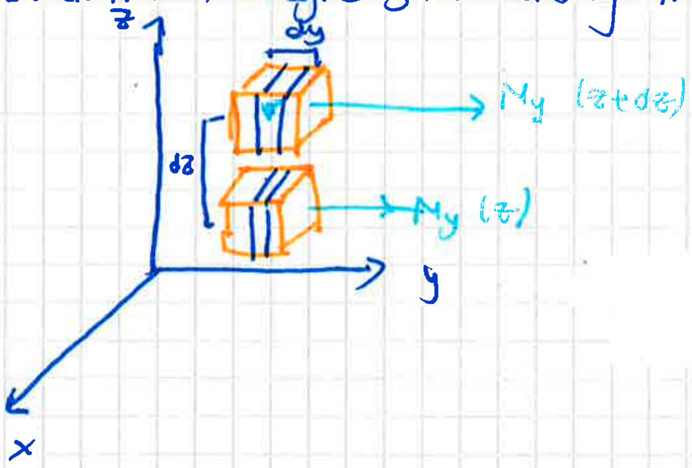
If we have a larger magnetization at y+dy compared to y we will get a net current in the x direction where the 2 slabs join given by:

$$I_x = \underbrace{[M_z(y+dy) - M_z(y)]}_{\text{this is just the difference in magnetization at the points } y \text{ and } dy} dz$$

$$= \frac{\partial M_z}{\partial y} dz dy$$

$$\Rightarrow \underbrace{\frac{I_x}{dz dy}}_{\substack{\text{volume current} \\ \equiv J_b}} = \frac{\partial M_z}{\partial y} \qquad \Rightarrow (J_b)_x = \frac{\partial M_z}{\partial y}$$

Similarly a non uniform magnetization along the z axis would contribute: $\frac{-\partial M_y}{\partial z}$.



Then considering both contributions:

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

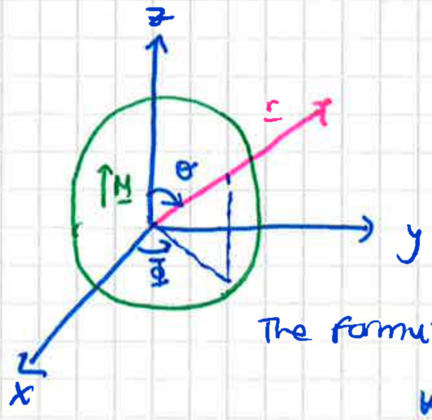
and in general:

$$\underline{J}_b = \nabla \times \underline{M}$$ showing that a non-uniform magnetization gives rise to a bound volume current.

It is evident that this current obeys the continuity equation ~~stated~~ for a steady current:

$$\nabla \cdot \underline{J}_b = 0$$ since $\underline{J}_b = \nabla \times \underline{M}$ is a curl, its divergence is trivially zero.

EXAMPLE: Find the magnetic field of a uniformly magnetized sphere.



We choose a coordinate system such that the z axis is aligned with the direction of the magnetization $\underline{M}$.

The goal is to calculate the bound currents ~~which will be~~

Using those bound currents as sources we can calculate then the magnetic field.

The formulas we have for the bound currents are:

$$\underline{K}_b = \underline{M} \times \hat{n}$$

$$\underline{J}_b = \nabla \times \underline{M}$$

The problem says we have a uniform magnetization, so in this case there will be no $\bar{J}_b$

$\therefore \underline{J}_b = 0$ because M is const in the sphere

So all we are left to do is calculate $\underline{K}_b = \underline{M} \times \hat{n}$

In this case we chose ~~define~~ a magnetization ∥ to the z axis so we can write it as:

$$\underline{M} = M\hat{k}$$

Now $\hat{n}$ is the unit vector that points normal to the surface of the sphere, in this case it's the radial direction $\hat{r}$, so $\hat{n}=\hat{r}$

So now we need to compute $\underline{M}\times\hat{n} = M\hat{k}\times\hat{r}$ We can either write $\hat{k}$ in spherical coord or write $\hat{r}$ in cartesian

We'll choose to write $\hat{k}=\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}$

$$\underline{M}\times\hat{n} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\Phi} \\ M\cos\theta & -M\sin\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = M\sin\theta\,\hat{\Phi} \qquad \therefore \underline{K}_b = M\sin\theta\,\hat{\Phi}$$

Now if we consider a ~~solid~~ sphere with a uniform surface charge σ that rotates at an angular velocity ω:

$$\underline{K} = \sigma\underline{v} = \sigma\omega R\sin\theta\,\hat{\Phi}$$

this surface current is equivalent to the bound current produced by magnetization, where $\underline{M}=\sigma\underline{\omega}R$

And last class we ~~derive~~ calculated $\underline{B}$ for such a sphere.

$$\underline{B} = \underbrace{\frac{2}{3}\mu_0\sigma R\underline{\omega}} = \frac{2}{3}\mu_0\underline{M} \quad \text{inside the sphere}$$