Summary of electrostatics + magnetostatics.

Lecture 15. The magnetic vector potential

Last time:

*Biot-Savart's law:

$$\underline{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\underline{s} \times \hat{r}}{r^2}$$ line of current

$$\underline{B} = \frac{\mu_0}{4\pi} \int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2} \, dA$$ surface current

$$\underline{B} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}) \times \hat{r}}{r^2} \, d\tau$$ Volume current

* Divergence & curl of $\underline{B}$:

$\nabla \cdot \underline{B} = 0$ Part of Maxwell's eqtns NON existence of magnetic monopoles

$\nabla \times \underline{B} = \mu_0 \underline{J}$ Ampère's law in differential form

$\Updownarrow$

$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$ Ampère's law in integral form

Today: Summary of magnetostatics, comparison w/ electrostatics

	Electrostatics	Magnetostatics
divergence	$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ Gauss' law	$\nabla \cdot \underline{B} = 0$ (no name)
curl	$\nabla \times \underline{E} = 0$ (no name)	$\nabla \times \underline{B} = \mu_0 \underline{J}$ Ampère's law

These equations are valid for electrostatics + magnetostatics, not complete Maxwell's equations but we are getting closer.

Let's have a close look at what they mean:

$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ Relates $\underline{E}$ (electric field) with the source charge density.

$\nabla \times \underline{E} = 0$ This is an equivalent statement to $\oint \underline{E} \cdot d\underline{s} = 0 \Leftrightarrow \underline{E}$ is a conservative field

These 2 equations contain the same information as Coulomb's law + the principle of superposition.

$$\left(\underline{F} = \frac{q}{4\pi\varepsilon_0} \int_V \frac{\rho \, dV}{r^2} \hat{r} \right.$$ Coulomb's law for volume charge distribution)

For magnetostatics:

①

$\nabla \cdot \underline{B} = 0$ Non existence of magnetic monopoles
Magnetic field lines do not begin or end anywhere. They typically form closed loops or extend to ∞.
i.e There are NO point sources for $\underline{B}$, i.e no magnetic analogue of electric charge.
$\Leftrightarrow$ magnetic fields are due to electric charges in motion.

$\nabla \times \underline{B} = \mu_0 \underline{J}$ Relates $\underline{B}$ and its sources $\underline{J}$

The sources of magnetic field are currents.

These equations are equivalent to Biot-Savart's law + principle of superposition.

$$\left(\underline{B} = \frac{\mu_0}{4\pi} \int_V \frac{\underline{J}(\underline{r}) \times \hat{r}}{r^2} \, dV \quad \text{Biot-Savart's law for volume current density} \right)$$

If we add to these equations the Lorentz force law:

$$\underline{F} = Q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

we have the equations that constitute a complete formulation of electrostatics & magnetostatics.

Some interesting facts:

→ Typically electric forces >> magnetic forces
- not intrinsic to the theory
- has to do with the values of ε_0 & μ_0
- In general only when the test charge and source charge are moving at $\sim c$ (speed of light)
  $|F_{elec}| \sim |F_{mag}|$

→ How do we notice magnetic forces then?
Sources of $\underline{B}$ are currents, a current is a # charges and velocity at which they are moving. If we add a "ton" of charges then even if they don't move too fast, we notice the effects.
These charges would also generate $\underline{E}$ to swamp $\underline{B}$, but if we keep the wire neutral by embedding the charges in an equal amount of opposite charge at rest, $\underline{E}$ cancels out & we can see $\underline{B}$ standing alone.

These sound like cumbersome arguments but it's actually what happens in a wire.

Magnetic vector potential

To simplify our life in electrostatics we defined the scalar potential φ:

$$\underline{E} = -\nabla\varphi, \quad \text{we could do this because} \quad \nabla \times \underline{E} = 0 \Rightarrow \nabla \times (\nabla\varphi) = 0 \ \forall\varphi$$

Can we do something similar for $\underline{B}$?

For the magnetic field we have that $\nabla \cdot \underline{B} = 0$

We know that $\nabla \cdot (\nabla \times \underline{f}) = 0$ for any $\underline{f}$, so we can define $\underline{A}$:

$$\boxed{\underline{B} = \nabla \times \underline{A}}$$ where $\underline{A}$ is the magnetic vector potential

We call it a vector potential by analogy to φ
but $\underline{A}$ is NOT connected to work or energy (Remember magnetic forces DON'T work)

$\underline{A}$ is NOT uniquely defined

In electrostatics: Given a charge distribution + ~~BCs to~~ boundary conditions $\Rightarrow \varphi$ is uniquely defined.

This was ~~actually~~ the uniqueness theorem we proved.

In magnetostatics: Does it work the same way? NO, there are an ∞ number of $\underline{A}$ corresponding to $\underline{B}$:

example: $\underline{B} = B_0 \hat{z}$. Find $\underline{A}$ that creates this $\underline{B}$:

$\underline{A}$. ~~curl~~ $\underline{B} = \nabla \times \underline{A}$:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = B_0$$

Here are a few possible solutions:

$$\underline{A} = -y B_0 \hat{x}$$

$$\underline{A} = x B_0 \hat{y}$$

$$\underline{A} = \frac{B_0}{2}(-y\hat{x} + x\hat{y})$$

$\underline{A}$ ... ∞ other solutions

Which one should we choose?

For convenience we will choose the one that also fulfills $\nabla \cdot \underline{A} = 0$. We will see why in a sec.

Poisson's equation for $\underline{A}$:

For electrostatics we obtained Poisson's equation by combining:

$$\underline{E} = \nabla \varphi \quad \text{and} \quad \nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}$$

For magnetostatics we can do something similar:

$$\underline{B} = \nabla \times \underline{A} \quad \text{and} \quad \nabla \times \underline{B} = \mu_0 \underline{J}$$

$$\nabla \times (\nabla \times \underline{A}) \underset{\uparrow}{=} \nabla(\nabla \cdot \underline{A}) - \nabla^2 A = \mu_0 \underline{J}$$

identity, pure algebra you can check

$$\Rightarrow \boxed{\nabla^2 A = -\mu_0 \underline{J}} \quad \text{if we choose} \quad \boxed{\nabla \cdot \underline{A} = 0} \quad \text{to simplify our life.}$$

Poisson's equation.

Rather, 3 Poisson's equations, one for each cartesian component.

Assuming that $\underline{J}$ goes to zero at infinity we have that the solution is:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r})}{r} dV$$

for volume current densities.

(see lecture #4 page 5 + lecture 5)

Briefly: Remember Poisson's eq:

$\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}$, which had a solution $\varphi(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} dV$

this is basically the same, but we have one of these equations for each cartesian component of A

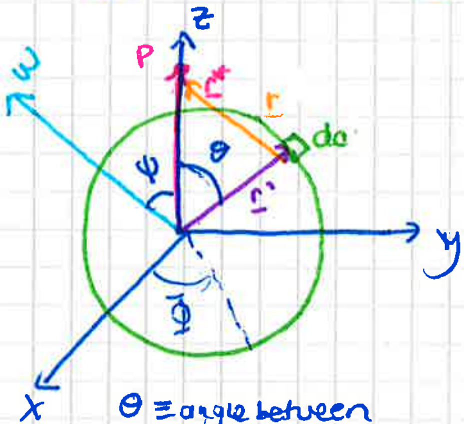
And for line currents we have:

$$\underline{A} = \frac{\mu_0}{4\pi}\int \frac{\underline{I}}{r}\, dl = \frac{\mu_0}{4\pi} \underline{I} \int \frac{1}{r}\, d\underline{l} \quad \text{line current}$$

$$\underline{A} = \frac{\mu_0}{4\pi}\int \frac{\underline{K}}{r}\, da \quad \text{surface current.}$$

Remember: r is the distance from the source pos vector to the point we are calculating the potential at.

EXAMPLE: A spherical shell of radius R, carrying a uniform surface charge σ, is spinning at angular velocity $\underline{\omega}$. Find the vector potential it produces at point $\underline{r}$.



$\theta \equiv$ angle between $\underline{r}$ and $\underline{r}'$

We will choose our coord system such that

$\underline{r}^*$ vector from the origin to the point we are calculating the potential

ω angular velocity. Axis of rotation of the sphere forms an angle ψ with respect to the z axis. ω lies on the xz plane.

$\underline{r}'$ vector from the origin to the patch da on the surface of the sphere with surface charge $d\sigma$

$\underline{r}$ vector from the source to the point we are calculating the potential at

With this we can set out to calculate $\underline{A}$:

$$\underline{A} = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r})}{r}\, da \quad (*)$$

In our case: ① $\underline{K}(\underline{r}) = \sigma \underline{v}$

σ ↑ surface charge on the sphere; $\underline{v}$ ↑ velocity of sphere

② $\underline{r} = \left[R^2 + r^2 - 2Rr\cos\theta\right]^{1/2}$ ↑ law of cosines

$c^2 = A^2 + B^2 - 2AB\cos\theta$

(remember in this case R is the radius of the sphere so $|\underline{r}'| = R$)

③ $da = R^2 \sin\theta\, d\theta\, d\Phi$ ← area element in spherical coord.

Now, what is $\underline{v}$?

Remembering from mechanics that the velocity of a point $\underline{r}'$ in a rotating rigid body is given by $\underline{\omega} \times \underline{r}'$, in this case:

$$\underline{v} = \underline{\omega} \times \underline{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta\cos\Phi & R\sin\theta\sin\Phi & R\cos\theta \end{vmatrix}$$

← First row is ω
← Second is $\underline{r}'$ in spherical coord.

definition of the $\times$ product: the first row are the components of ω, the second the components of r'

$$= R\omega\left[-(\cos\psi\sin\theta\sin\Phi)\hat{x} + (\cos\psi\sin\theta\cos\Phi - \sin\psi\cos\theta)\hat{y} + (\sin\psi\sin\theta\sin\Phi)\hat{z}\right]$$

this is the only term that survives

— these will be zero

We will need to integrate this, but don't despair. If we notice:

All the terms except for ~~the~~ one ~~these~~ involve either $\sin\Phi$ or $\cos\Phi$. Since:

$$\int_0^{2\pi} \sin\Phi \, d\Phi = \int_0^{2\pi} \cos\Phi \, d\Phi = 0$$ these terms will contribute nothing to the integral so we are left only with one term.

Then subs $\underline{K} = \sigma \underline{v}$ into ⊛, considering that only $-\sin\psi \cos\theta \, \hat{y}$ will survive:

$$\underline{A} = -\frac{\mu_0 \sigma R \omega \sin\psi}{4\pi} \int_0^{2\pi}\int_0^{\pi} \frac{\cos\theta \, \overbrace{R^2 \sin\theta \, d\theta \, d\Phi}^{da}}{\underbrace{[R^2 + r^2 - 2Rr\cos\theta]^{1/2}}_{r}} \, \hat{y}$$

$$= -\frac{\mu_0 \sigma R^3 \omega \sin\psi \, (2\pi)}{4\pi} \int_0^{\pi} \frac{\cos\theta \sin\theta \, d\theta}{[R^2 + r^2 - 2Rr\cos\theta]^{1/2}} \, \hat{y}$$

To solve the remaining integral we set $u \equiv \cos\theta$ ~~and~~ so it becomes:

$$\int_{-1}^{1} \frac{u \, du}{[R^2 + r^2 - 2Rru]^{1/2}} = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \left[R^2 + r^2 - 2Rru\right]^{1/2} \Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2} \left[(R^2 + r^2 + Rr) \underbrace{|R - r|} - (R^2 + r^2 - Rr)(R + r) \right]$$

depending of where is r this will ~~...~~ ~~actually~~ be positive or negative

For $R > r$ the expression above reduces to: $\dfrac{2r}{3R^2}$

a point inside the sphere

For $R < r$ the expression above reduces to: $\dfrac{2R}{3r^2}$

a point outside the sphere

Then ~~...~~

$$\underline{A}(\underline{r}) = \begin{cases} -\dfrac{\mu_0 R^3 \omega \sin\psi \, \sigma}{2} \left(\dfrac{2r}{3R^2} \right) \hat{y} & \text{for } r < R \\[2ex] -\dfrac{\mu_0 R^3 \omega \sin\psi \, \sigma}{2} \left(\dfrac{2R}{3r^2} \right) \hat{y} & \text{for } r > R \end{cases}$$

and noting that $\underline{\omega} \times \underline{r} = -\omega r \sin\psi \, \hat{y}$ we can write the above as:

$$\underline{A}(\underline{r}, \theta, \Phi) = \begin{cases} \dfrac{\mu_0 R \sigma}{3} (\underline{\omega} \times \underline{r}) \, \hat{\Phi} & \text{for } r < R \\[2ex] \dfrac{\mu_0 R^4 \sigma}{3r^3} (\underline{\omega} \times \underline{r}) \, \hat{\Phi} & \text{for } r > R \end{cases}$$

Out of curiosity we can calculate the field inside the sphere and see that:

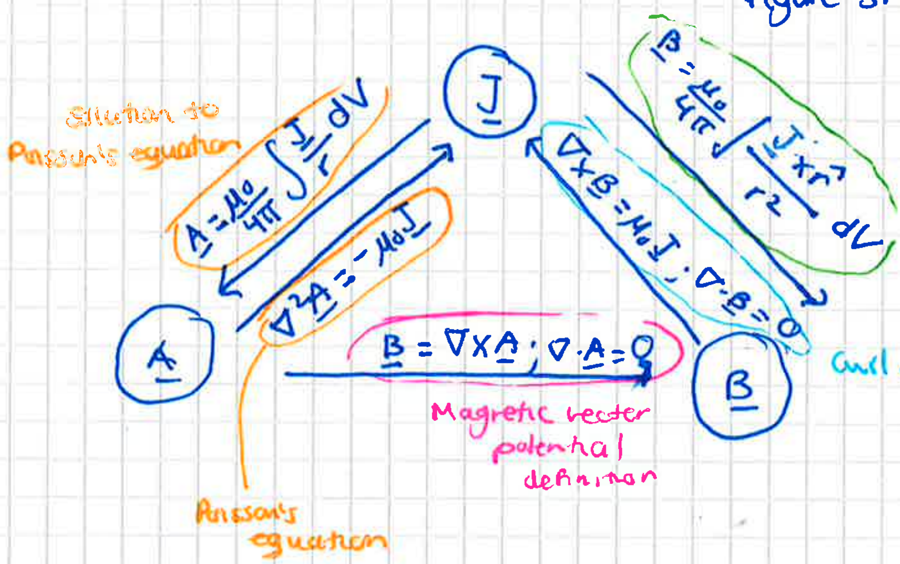
$$\underline{B} = \nabla \times \underline{A} = \frac{2\mu_0 R \omega \sigma}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= \frac{2}{3}\mu_0 \sigma R \omega \hat{z}$$

$$= \frac{2}{3}\mu_0 \sigma R \underline{\omega}$$ uniform inside the sphere.

Now that we have defined the vector potential we can write a triangle to summarize the relationships between sources, field and potential. Analogous to what we did for electrostatics.

Figure 5.48 Griffiths

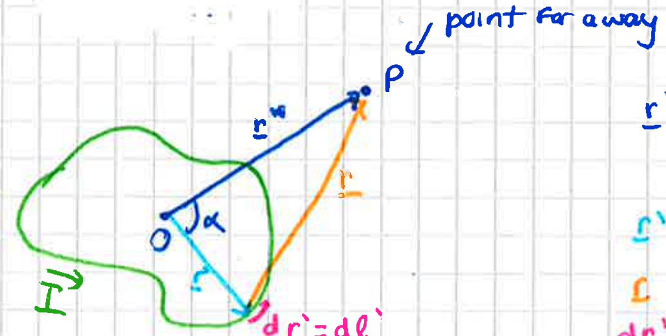


Multipole expansion of the magnetic vector potential

Similar to what we did for the electric potential, if we want to approximate $\underline{A}$ due to a localized current distribution, at a point far away from it, we can use a multipole expansion.

The general idea is to write the potential as a power series of $\frac{1}{r}$, where r is the distance to the point in question. If r is large, then the expansion will be dominated by the first non-vanishing terms.

Our goal is to expand the expression for A in a series of 1/r terms:



$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{r} d\underline{\ell}' \quad (*)$$

To clarify the diagram:

$\underline{r}^*$ vector from origin to point we want the potential at, $\underline{r} >> \underline{r}'$ so it's far away

$\underline{r}'$ vector from origin to the source of $\underline{A}$, in this case the source is $d\underline{\ell}'$

$\underline{r}$ vector from source to pt at which we calculate $\underline{A}$.

$d\underline{r}' = d\underline{\ell}'$ source

I current flowing in loop

α angle between $\underline{r}'$ and $\underline{r}^*$

Now that we have all these precisions, we need to expand the expression for the potential given in equation ⊛.

As we found in lecture 8 we can expand 1/r as:

$$\frac{1}{r} = \frac{1}{[r^2+(r')^2-2rr'\cos\alpha]^{1/2}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\alpha)}_{\text{gen funct for legendre polynomials}}$$

Using this expression we can rewrite equation ⊛ as:

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\,d\underline{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\,d\underline{l}'$$

or more explicitly:

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\underline{l}'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\,d\underline{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\underline{l}'}_{\text{quadrupole}} + \dots\right]$$

Let's calculate a couple of terms.

monopole $\oint d\underline{l}' = 0$

integral of a displacement around a closed loop

are you surprised by this?

We derived last week that $\nabla\cdot\underline{B} = 0$
$\Rightarrow \nexists$ magnetic monopoles.
we used this equation to define $\underline{A}$,
so this is ~~all~~ as it should be.

dipole $\underline{A}_{dip}(\underline{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\,d\underline{l}' = \frac{\mu_0 I}{4\pi r^2}\oint(\hat{r}\cdot\underline{r}')\,d\underline{l}'$

definition of dot product

We can re-write this integral using the identity
$\oint(\underline{c}\cdot\underline{r})\,d\underline{l} = \underline{a}\times\underline{c}$ with $\underline{c} = \hat{r}$ so then $\oint(\hat{\underline{r}}\cdot\underline{r}')\,d\underline{l}' = -\hat{r}\times\int d\underline{a}'$

where $\underline{a}$ is the area vector
$\underline{a} \equiv \int d\underline{a}$

then the dipole ~~term~~ contribution to $\underline{A}$ becomes:

$$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{I}{r^2}\left[-\hat{r}\times\int d\underline{a}'\right] = \frac{\mu_0}{4\pi}\frac{1}{r^2}\underbrace{\left(I\int d\underline{a}'\right)}_{m}\times\hat{r}$$

$$\therefore \boxed{\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{m}\times\hat{r}}{r^2}}$$

where we have defined

$$\boxed{\underline{m} \equiv I\int d\underline{a} = I\underline{a}}$$

magnetic dipole moment

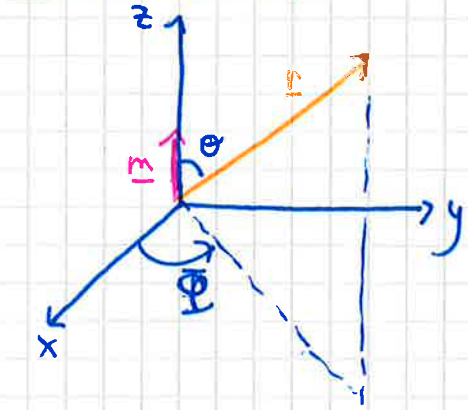
here $\underline{a}$ is the "vector" area of the loop. If the loop is flat $\underline{a}$ is the ordinary area enclosed. Direction of area is given by right hand rule, where fingers point in direction of current.

From the equation of the dipole moment, we can see that it is INdependent of the choice of coordinate system. This is different from the electric dipole moment that we saw would depend on the choice of ~~coordinate system~~ origin.

For the case of electric dipoles, ~~the~~ choice of origin didn't matter if the total charge was zero. So it's not so surprising that for magnetic dipoles the choice of origin doesn't matter. After all we don't have magnetic monopoles.

In practice the dipole potential is a ~~good~~ suitable approximation whenever the distance r greatly exceeds the size of the loop.

Field of a dipole

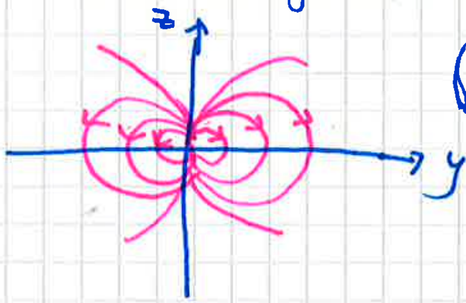


We just derived that for a dipole:

$$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\Phi}$$

$$\Rightarrow \underline{B}(\underline{r}) = \nabla\times\underline{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

which looks very similar to an electric dipole

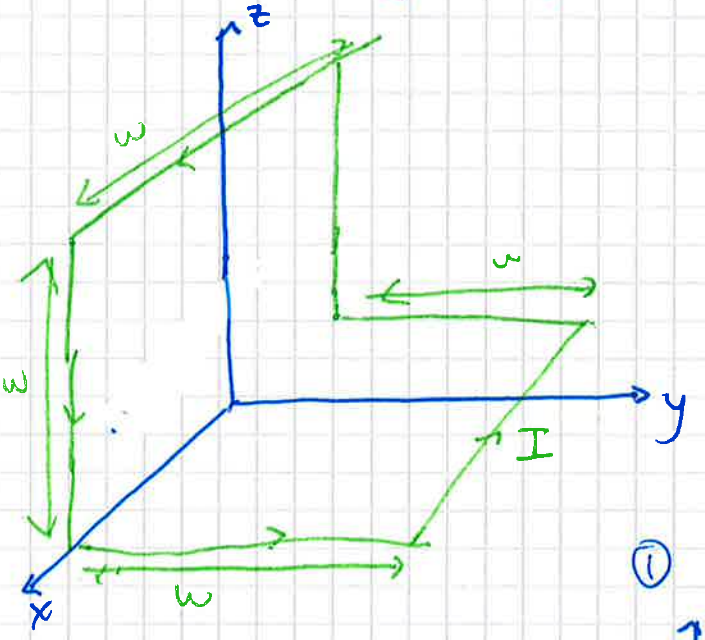


$$\left(\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left(\sin\theta\,\hat{\theta} + 2\cos\theta\,\hat{r}\right)\right)$$

In a coordinate free form we can write

$$\boxed{\underline{B}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\underline{m}\cdot\hat{r})\hat{r} - \underline{m}\right]}$$

EXAMPLE: Find the magnetic dipole moment of the "bookend-shaped" loop shown below:



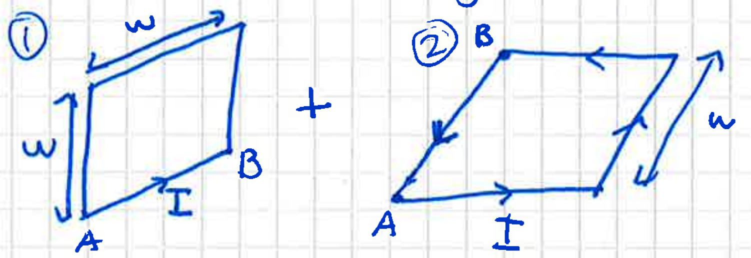
All the sides of the loop have length w. There is a current I going through the loop.

The goal is to calculate the dipole moment we just derived.

$$\underline{m} = I\underline{a}$$

The easiest is to consider this loop as the superposition of two plane square loops.

The "extra" sides AB will cancel when we put them together:



The magnetic dipole moment is

$$\underline{m} = \underbrace{I\,w^2\hat{y}}_{\text{horizontal loop (2)}} + \underbrace{Iw^2\hat{z}}_{\text{vertical loop (1)}}$$

its magnitude is $\sqrt{2}\,Iw^2$ and points along 45° line $z=y$