Lecture 11. Electric currents, ohm's law

Last time:

* The electric displacement:

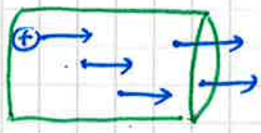
$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$$ field due to $\underline{E}$ + polarization of a dielectric

* Gauss's law for dielectrics:

$$\nabla \cdot \underline{D} = \rho_{free}$$ differential form

$$\oint \underline{D} \cdot d\underline{a} = Q_{free}$$ integral form

TODAY: Electric currents



e.g. conductor

region where charges flow
the electric current is the rate at which charges flow across a cross sectional area

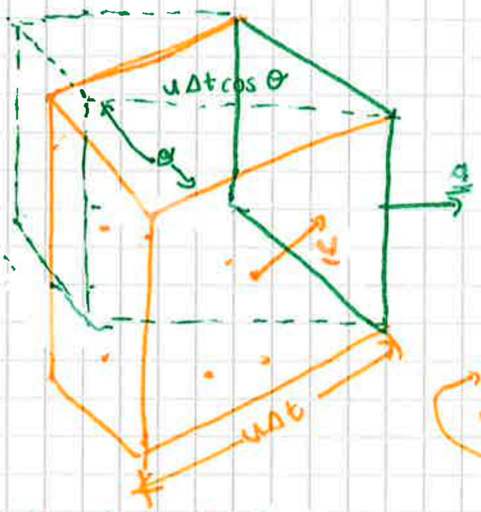
$$\boxed{I \equiv \frac{dQ}{dt}}$$ charge / unit time flowing through a surface

Units:
$[I]$ = ampere $\equiv 1 \dfrac{\text{coulomb}}{\text{second}}$

lightning ~ Mega amperes
biological currents ~ nano amperes

Flow has a direction, the convention used:
Direction of flow of ⊕ charges is the direction of flow of current
$\Leftrightarrow$
Direction of flow of ⊖ charges is opposite direction of current.

Current density
To relate current, a macroscopic quantity with the movement of microscopic charges we need to define a density.



This prism has a base area a and an altitude uΔt cos θ so its volume is $a u \Delta t \cos\theta$ or $\underline{a} \cdot \underline{u}\, \Delta t$

If we look at a frame with area a, particles that will pass through it in Δt are the ones contained in the prism

What is the current density through area $\underline{a}$?

$$I = \frac{\Delta Q}{\Delta t} = \frac{q\,\Delta N}{\Delta t} = \frac{q n V}{\Delta t} = \frac{q n (a \cos\theta\, u \Delta t)}{\Delta t}$$

ΔN = # charges n = # charges / unit volume

$$= q n \underline{u} \cdot \underline{a}$$

we're not done yet.
we can have # particles passing through the cube i.e $q \neq$
and moving at ≠ velocities i.e $u \neq$
so to generalize this:

$$I = n_1 q_1 \underline{u}_1 \cdot \underline{a}_1 + n_2 q_2 \underline{u}_2 \cdot \underline{a}_2 + \dots = \underline{a} \cdot \underbrace{\sum_k n_k q_k \underline{u}_k}_{\equiv \underline{J} \text{ current density}}$$

$$\boxed{\underline{J} = \sum_k n_k q_k \underline{u}_k \equiv \sum_k \rho_k \underline{u}_k}$$

where $\rho_k = n_k q_k$ volume charge density

$[J] = \frac{\text{amperes}}{m^2}$

now if we consider that not all charges have the same velocity:

average velocity $\langle \underline{u}_k \rangle = \frac{1}{N_k} \sum_i (\underline{u}_k)_i$ and with this we have that:

$$\underline{J} = \sum_k q_k n_k \langle \underline{u}_k \rangle = \sum_k \rho_k \langle \underline{u}_k \rangle$$

and for an arbitrary surface S :

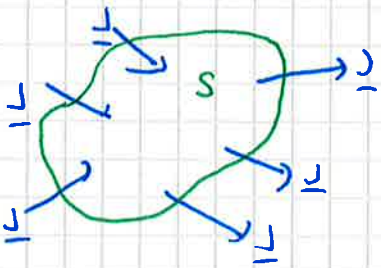
$$\boxed{I = \int_S \underline{J} \cdot d\underline{a}}$$

Typically we think of currents as electrons moving in a conductor (e.g. wire). but as long as we have moving charges we have a current.

Demo 646 (related to past demo, are you a conductor)

Ions in solutions can also make a current.

The continuity equation



we have a closed surface S. some charge enters & some charge exits.

what happens to the charge? → pile up inside the surface / leave the surface

$$\underbrace{\oint_S \underline{J} \cdot d\underline{a}}_{\substack{\text{rate at which charge is leaving the} \\ \text{volume (flux through S)}}} = \frac{-\partial Q_{inside}}{\partial t} = -\frac{\partial}{\partial t} \int_V \rho \, dV$$

Now we can apply Gauss's theorem to the left side of the equation to get:

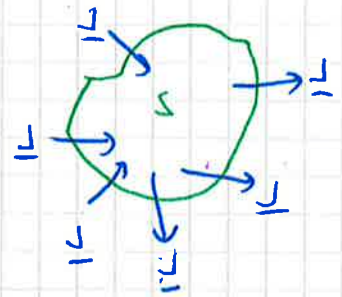
$$\oint_S \underline{J} \cdot d\underline{a} = \oint_V \nabla \cdot \underline{J} \, dV \Rightarrow \int_V \nabla \cdot \underline{J} \, dV = -\frac{\partial}{\partial t} \int_V \rho \, dV$$

divergence theorem

$$\Rightarrow \int_V \left(\nabla \cdot \underline{J} + \frac{\partial}{\partial t} \rho \right) dV = 0$$

$$\Rightarrow \boxed{\nabla \cdot \underline{J} + \frac{\partial}{\partial t} \rho = 0} \text{ continuity equation}$$

Thoughts on the continuity equation:



$\nabla \cdot \underline{J} + \frac{\partial}{\partial t}\rho = 0$

What does it teach us?

* electric charge is conserved in the presence of currents

* For steady currents (i.e. $\frac{\partial \rho}{\partial t} = 0$)

this means that there is NO accumulation of charge inside the surface

$$\boxed{\nabla \cdot \underline{J} = 0 \quad \leftarrow \text{For } \underline{\text{steady}} \text{ currents ONLY}}$$

Microscopic Ohm's law

We know that electric fields cause charges to move.
In most substances & over a wide range of electric field strengths:

$$\boxed{\underline{J} = \sigma \underline{E}}$$

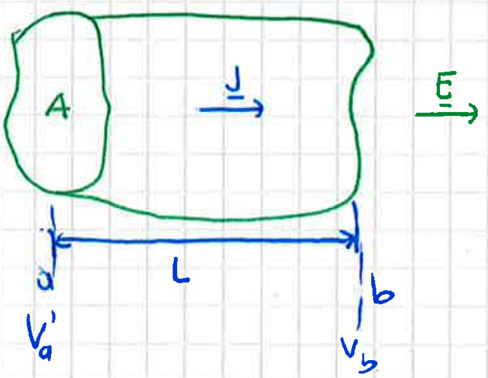
current density is proportional to electric field strength

$\sigma \equiv$ conductivity of the material

wait, wasn't $\underline{E} = 0$ inside a conductor?
Yes, if we ~~do~~ just let charges flow until they settle down, in some configuration that cancels an applied field internally
Here we are considering a situation where charges don't pile up e.g. they don't settle down.

Ohm's law is an empirical law, i.e a generalization derived from experiments
this is not a theorem that should be universally obeyed but it turns out it's really useful so now we'll work through its consequences.

Macroscopic Ohm's law



* We have a uniform material of length L
* The material is in a uniform electric field $\underline{E}$ || L

What is the potential difference between the two ends? $V = EL$, why? $\Delta V = V_b - V_a = -\int_a^b \underline{E} \cdot d\underline{s} = EL$

Let's consider ohm's microscopic law:

$J = \sigma E$

by definition $J = \frac{I}{A}$ and $V = EL \Rightarrow E = \frac{V}{L}$ subs these into ohm's microscopic law:
$\uparrow$ area

$$\frac{I}{A} = \frac{\sigma V}{L} \Rightarrow V = \frac{IL}{A\sigma} \quad \boxed{R = \frac{L}{\sigma A}}$$

$$\boxed{\text{then } V = IR} \quad \text{with } R = \frac{L}{\sigma A}$$

because this is a phenomenological law it's not universally obeyed. If this is true for a material, it's called Ohmic, if it's not then the material is non-Ohmic.

Resistance

$$R \equiv \frac{L}{\sigma A} \equiv \frac{\rho L}{A}$$; Here ρ is the resistivity, which is the inverse of the conductivity.

* Proportionality constant between V and R on Ohm's law
* Units

$$[R] = \Omega \quad \text{ohm} = \frac{\text{Volt}}{\text{Ampere}}$$

* It depends on the geometry, it has L & A so it's inversely prop to A & prop to L
* Depends on the material, it has σ so it's inversely prop to the conductivity

Demo 526 Ohm's law | Demo 618 Resistance in conductors

Resistivity

$\rho = \frac{1}{\sigma}$, so the microscopic Ω's law reads: $\underline{J} = \left(\frac{1}{\rho}\right)\underline{E}$

* Reciprocal of the conductivity. Describes how fast electrons can travel in a material.
* By convention/custom ρ is used as the symbol for resistivity. This is confusing since we use ρ for vol charge densities, but it's the custom.
* Depends on the chemistry of the material & the temperature, etc.

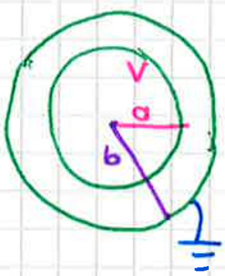
in terms of resistivity the macroscopic Ω's law reads:

$$\boxed{R = \frac{\rho L}{A}}$$ where ρ is the resistivity

Demo : ~~7080~~ ~~measurement~~ ~~4888~~ 444 Resistivity depends on temperature

Room temp → more collisions → ρ increases
low temp → mean free path of e⁻ dominated by impurities in the material.

EXAMPLE: Resistance of a spherical shell



We have 2 concentric spheres of radii a & b

Outside sphere is grounded → V=0
Inside sphere is at a potential V
⇒ $\varphi_{inner} = V$
$\varphi_{outer} = 0$

There is a potential difference so there is a current

Q. What is the resistance R?

Important formulas we just derived:

$I = \int_S \underline{J} \cdot d\underline{a}$ — current through any surface S

$\underline{J} = \sigma \underline{E}$ — current density is proportional to electric field strength (microscopic Ω's law)

$V = RI$ — macroscopic Ω's law

We will use these expressions to solve the problem.

1. We will use first $\underline{J} = \sigma \underline{E}$ to calculate the current density.

We have information on φ, so we should use this to calculate $\underline{E}$.
We know that for spherical symmetry

$$\varphi(\underline{r}) = A + \frac{B}{r}$$ where A & B are constants (see lecture 7)

we are given the boundary conditions $\varphi(a) = V$ and $\varphi(b) = 0$

$$\varphi(a) = A + \frac{B}{a} = V$$

$$\varphi(b) = A + \frac{B}{b} = 0 \Rightarrow A = -\frac{B}{b}$$ (*) subs this in the 1st eq:

$$-\frac{B}{a} + \frac{B}{b} = V \Rightarrow -B\left(\frac{b-a}{ab}\right) = V$$

$$\Rightarrow B = \frac{-V(ab)}{b-a}$$ so putting this back into (*):

$$A = \frac{1}{\cancel{b}}\left(\frac{+V a\cancel{b}}{b-a}\right)$$ and subs this into the eq for $\varphi(r)$

$$\varphi(r) = \frac{+aV}{b-a} + \frac{1}{r}\left(\frac{-V(ab)}{b-a}\right) = V\left[\frac{-a}{b-a} + \frac{1}{r}\frac{ab}{b-a}\right]$$

Now that we have φ we can calculate $\underline{E}$:

$\underline{E} = -\nabla\varphi$ so in this case:

$$\underline{E} = -\frac{\partial\varphi}{\partial r} = \frac{V}{r^2}\frac{ab}{b-a}\,\hat{r}$$

so now that we have $\underline{E}$ we can finally get:

$$J = \sigma\underline{E} = \sigma V \frac{ab}{b-a}\frac{1}{r^2}$$

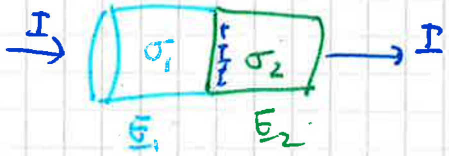
2. Now we will use the expression to get the current through the sphere

$$I = \int_{sphere} \underline{J}\cdot d\underline{a} = \underline{J}\cdot\underbrace{\int_{sphere} d\underline{a}}_{area = 4\pi r^2} = \sigma V\frac{ab}{b-a}\frac{1}{\cancel{r^2}}\left(4\pi \cancel{r^2}\right) = 4\pi\sigma V\frac{ab}{b-a}$$

3. Now that we have the current we can calculate the resistance using the expression for the macroscopic ohm's law:

$$V = RI \Rightarrow R = \frac{V}{I} = \frac{V}{4\pi\sigma V\left(\frac{ab}{b-a}\right)} = \frac{b-a}{4\pi\sigma ab}$$

EXAMPLE! How much charge builds up at the junction of two materials w/ different conductivity.
What if σ is not constant?



If a wire is made of 2 conductors I flowing through the cylinder should be the same

$$I = A\sigma_1 E_1 = A\sigma_2 E_2 \quad (\star)$$

This is for a steady state of current flow

where we have used the definition $J = \frac{I}{A} = \sigma E \Rightarrow I = \sigma E A$

Equation $(\star)$ implies that:

$$E_2 = \left(\frac{\sigma_1}{\sigma_2}\right) E_1$$

We will now use gauss's law to obtain the charge built up at the interface:

$$\int \underline{E} \cdot d\underline{A} = \frac{Q_{int}}{\varepsilon_0} \Rightarrow \int \underline{E} \cdot d\underline{A} = (E_2 - E_1) A = \frac{Q_{int}}{\varepsilon_0}$$

And substituting the expression above for E_2:

$$A\varepsilon_0 \left(\frac{\sigma_1}{\sigma_2} E_1 - E_1\right) = Q_{int} \Rightarrow \varepsilon_0 A E_1 \left(\frac{\sigma_1}{\sigma_2} - 1\right) = \varepsilon_0 \underbrace{A\sigma_1 E_1}_{I} \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) = Q_{int}$$

and expressing this in terms of I (from eq $(\star)$):

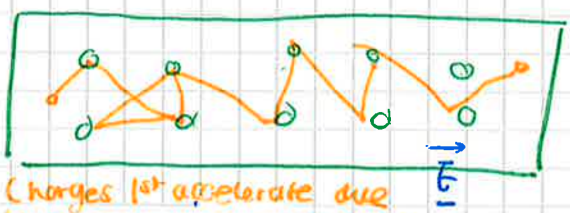
$$Q_{int} = \varepsilon_0 I \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right)$$

Thoughts on Ohm's law (Physical intuition)

*Ohm's law in its microscopic formulation:

$\underline{J} = \sigma \underline{E}$ or in plain English: "A constant electric field creates a steady current".

$\underline{E} \propto \underline{v}$ where $\underline{v}$ is the velocity at which the charges move



Charges 1st accelerate due to $\underline{E}$, then bump into nuclei & are scattered. → the average behavior is a uniform drift

In a material:
Charges are moving in an effectively viscous medium.

Just like a skydiver in free fall they first accelerate, then reach constant velocity.

$$F = q\underline{E} - \underbrace{m\gamma \underline{v}}_{\text{'friction w/ medium'}} = m\frac{d\underline{v}}{dt}$$

Forces on a charge moving

When the charges reach terminal velocity:

$$\frac{d\underline{v}}{dt}=0 \Rightarrow q\underline{E} = -m\gamma \underline{v} \Rightarrow v = -\frac{qE}{m\gamma}$$ ← terminal velocity at which charges move in the material where γ is a friction coefficient

which gives us another way to express J.

From the beginning of the lecture:

$$\underline{J} = \sum_k n_k q_k \underbrace{\langle \underline{u}_k \rangle}_{\uparrow \text{ this is } v} = \frac{n_+ q_+^2 E}{m_+\gamma_+} + \frac{n_- q_-^2 \underline{E}}{m_-\gamma_-} = \underbrace{\left(\frac{n_+ q_+^2}{m_+\gamma_+} + \frac{n_- q_-^2}{m_-\gamma_-}\right)}_{\sigma} \underline{E}$$

so we recover ohm's law.

We can also calculate the drift velocity in terms of the average time travelled to have a more useful expession for a material.

So again we have $\underline{J} = nq\underline{v}$ and we are interested in finding an expression for $\underline{v}$:

The forces on an electron in a conductor are:

$\underline{F}_e = -e\underline{E}$ due to the electric field which gives it an acceleration:

$$\underline{a} = \frac{F_e}{m_e} = \frac{-e\underline{E}}{m_e}$$

Let the velocity of an electron immediately after a collision be $\underline{v}_i$, so then the velocity of the electron immediately before the next collision is:

$$\underline{v}_f = \underline{v}_i + \underline{a}t = \underline{v}_i - \frac{e\underline{E}}{m_e}t$$ where t is the time travelled

The average $\underline{v}_f$ over all time intervals is:

$$\langle \underline{v}_f \rangle = \langle \underline{v}_i \rangle - \frac{e\underline{E}}{m_e}\langle t \rangle = \underline{v}$$
↑ drift velocity we wanted to find.

in the absence of an electric field the vel of an electron is completely random $\Rightarrow \langle \underline{v}_i \rangle = 0$. If we define $\tau = \langle t \rangle$ as the average characteristic time between collisions (the mean free time) we have:

$$\underline{v} = \langle \underline{v}_f \rangle = \frac{-e\underline{E}}{m_e}\tau$$ so subs this in the eq. for the current density we have:

$$\underline{J} = -ne\underline{v} = -ne\left(\frac{-e\underline{E}}{m_e}\tau\right) = \frac{ne^2\tau}{m_e}\underline{E}$$

$$\boxed{\underline{J} = \frac{ne^2\tau}{m_e}\underline{E}}$$ useful expression to solve some problems

EXAMPLE: Calculation of drift velocity

The resistivity of sea water is $\sim 25\,\Omega\cdot cm$. The charge carriers ~~are~~ are Na^+ and Cl^- ions, of each there are $\sim 3\times10^{20}/cm^3$. If we fill a 2m long plastic tube with sea water and connect a 12V battery to electrodes at each end, what is the average drift velocity of the ions in cm/s?

The current in a conductor of cross sectional area A is related to the drift speed ~~by~~ of the charge carriers by:

$$J = +nev_d \Rightarrow I = enAv_d$$

where n is the ~~#~~ of charges / unit volume.

using: $I = JA$

We can then re-write Ohm's law as:

$$V = RI = (neAv_d)\underbrace{\left(\frac{\rho l}{A}\right)} = nev_d\rho l \Rightarrow v_d = \frac{V}{ne\rho l}$$

this is the definition of resistence we derived today

Now substituting the values we get from the problem:

$$v_d = \frac{12V}{(6\times10^{20}/cm^3)\underbrace{(1.6\times10^{-19}C)}(25\,\Omega cm)(200\,cm)}$$

e = elementary charge value

because we have $3\times10^{20}/cm^3$ of Na and $3\times10^{20}/cm^3$ of Cl

$$= 2.5\times10^{-5}\frac{V\,cm}{C\,\Omega} = 2.5\times10^{-5}\frac{cm}{s}$$