Lecture 23. Energy and momentum of EM waves

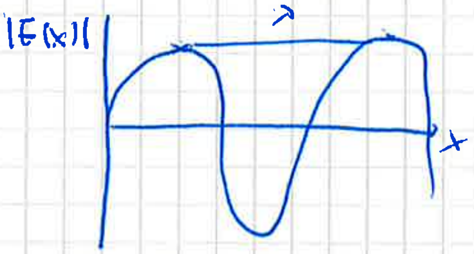
Last time:

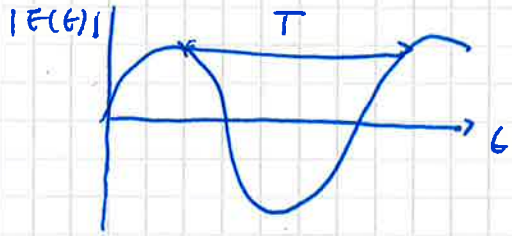
* EM plane waves:
  - ↳ Planar EM waves are transverse → direction of $\underline{E}$ & $\underline{B}$ is $\perp$ to direction of propagation $\underline{k}$
  - ↳ propagate at the speed of light c
  - ↳ The relationship between the magnitudes of $\underline{E}$ & $\underline{B}$ is:

$$E_0 = \pm c B_0$$

  - ↳ Can be referred to as monochromatic, different frequencies will correspond to different colors.

* For a wave the following relations hold: $\underline{E} = E_0 \sin(kx - \omega t)$





$\underline{E} = E_0 \sin(k_x x)$ $\qquad$ $\underline{E} = E_0 \sin(\omega t)$

$$\omega = \frac{2\pi}{T} = 2\pi\nu \quad ; \quad \omega = ck \quad ; \quad \lambda\nu = c \quad \text{where } \nu \equiv \text{frequency} = \frac{1}{T}$$

* Polarization of EM waves:
  - ↳ linearly polarized → directions of $\underline{E}_0$ & $\underline{B}_0$ are constant in time
  - ↳ circularly polarized → direction of $\underline{E}_0$ & $\underline{B}_0$ rotate at a frequency ω.

Today: Electromagnetic energy

We have already discussed that the electric & magnetic fields store energy.

Suppose we have some charge & current configuration which at time t produces the fields $\underline{E}$ & $\underline{B}$

In the next instant dt the charges move around a bit

How much work, dW, is done by the EM forces acting on these charges in the time interval dt?

According to Lorentz's force law:

$$\underbrace{\underline{F} \cdot d\underline{l}}_{dW} = q(\underline{E} + \underline{v} \times \underline{B}) \cdot \underbrace{\underline{v}\, dt}_{\text{this is just } d\underline{l}} = q\underline{E} \cdot \underline{v}\, dt + \underbrace{\underline{v} \times \underline{B} \cdot \underline{v}\, dt}_{=0 \text{ because}} = q\underline{E} \cdot \underline{v}\, dt \; *$$

=0 because $\underline{v} \times \underline{B} \perp$ to $\underline{v}$ and the dot prod of 2 $\perp$ vectors is zero

We can rewrite this last equation in terms of the charge and current densities ρ and $\underline{J}$:

Remembering $q \to \int \rho \, d\tau$

$\rho \underline{v} \to \underline{J}$, equation * becomes:

$$dW = \rho \, d\tau \, \underline{E} \cdot \frac{\underline{J}}{\rho} dt = \underline{E} \cdot \underline{J} \, d\tau \, dt \Rightarrow \frac{dW}{dt} = \int_V \underline{E} \cdot \underline{J} \, d\tau$$

From the last equation we can see that $\underline{E}\cdot\underline{J}$ is the work per unit time per unit volume i.e the power delivered per unit volume. We would like to express this quantity in terms of $\underline{E}$ & $\underline{B}$ only so we will use Maxwell's-Ampère's law to rewrite $\underline{J}$:

Remembering: $\nabla\times\underline{B} = \mu_0\underline{J} + \mu_0\varepsilon_0\frac{\partial\underline{E}}{\partial t}$

$$\Rightarrow \frac{\nabla\times\underline{B}}{\mu_0} - \frac{\mu_0\varepsilon_0}{\mu_0}\frac{\partial\underline{E}}{\partial t} = \underline{J}$$

We can rewrite ~~[illegible]~~ $\underline{E}\cdot\underline{J}$ as:

$$\underline{E}\cdot\underline{J} = \frac{1}{\mu_0}\underline{E}\cdot(\nabla\times\underline{B}) - \varepsilon_0\underline{E}\cdot\frac{\partial\underline{E}}{\partial t}$$

And from product rule 6 from Griffiths:

$$\nabla\cdot(\underline{E}\times\underline{B}) = \underline{B}\cdot\underbrace{(\nabla\times\underline{E})}_{-\frac{\partial\underline{B}}{\partial t}\ \text{(Faraday's law)}} - \underline{E}\cdot(\nabla\times\underline{B}) = -\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} - \underline{E}\cdot(\nabla\times\underline{B})$$

~~Therefore we see that~~: $\Rightarrow \nabla\cdot(\underline{E}\times\underline{B}) + \underline{B}\cdot\frac{\partial\underline{B}}{\partial t} = -\underline{E}\cdot(\nabla\times\underline{B})$

$$\Rightarrow -\frac{1}{\mu_0}\nabla\cdot(\underline{E}\times\underline{B}) - \frac{1}{\mu_0}\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} = \frac{\underline{E}\cdot(\nabla\times\underline{B})}{\mu_0}$$

The the equation for $\underline{E}\cdot\underline{J}$ becomes:

$$-\frac{1}{\mu_0}\nabla\cdot(\underline{E}\times\underline{B}) - \frac{1}{\mu_0}\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} - \varepsilon_0\underline{E}\cdot\frac{\partial\underline{E}}{\partial t} = \underline{E}\cdot\underline{J}$$

$$\Rightarrow \underline{E}\cdot\underline{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\underline{E}\times\underline{B})$$

Now putting this into the equation for dW/dt:

$$\frac{dW}{dt} = \int_V(\underline{E}\cdot\underline{J})\,d\tau = -\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\underbrace{\int_V\nabla\cdot(\underline{E}\times\underline{B})\,d\tau}_{=\oint_S(\underline{E}\times\underline{B})\cdot d\underline{a}\ \text{using the divergence theorem}}$$

$$\therefore \boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_S(\underline{E}\times\underline{B})\cdot d\underline{a}}$$

where S is the surface bounding the volume V

Poynting's theorem

"work-energy theorem of electrodynamics"

What is the physical interpretation of this equation?

$$\frac{dW}{dt} = \underbrace{-\frac{d}{dt}\int\frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)d\tau}_{\text{energy stored in the fields}} \underbrace{-\frac{1}{\mu_0}\oint_S(\underline{E}\times\underline{B})\cdot d\underline{a}}_{\text{looks like flux}}$$

$\rightarrow$ rate at which energy is transported out of the volume V across its boundary surface by the EM fields.

Remember:

$$W_e = \frac{\varepsilon_0}{2}\int E^2\,d\tau \qquad W_m = \frac{1}{2\mu_0}\int B^2\,d\tau \quad \therefore u = \frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)$$

energy/unit volume

So What does this equation say?

work done on the charges by the EM force = decrease in energy stored in the remaining fields − energy that has flowed out of the surface.

~~[illegible]~~

And the energy per unit time, per unit area, transported by the fields is called:

$$\boxed{\underline{S} \equiv \frac{1}{\mu_0}(\underline{E} \times \underline{B})} \quad \text{Poynting vector}$$

Specifically $\underline{S} \cdot d\underline{a}$ is the energy per unit time crossing the infinitesimal surface $d\underline{a}$ ⇒ the energy flux, so we can interpret $\underline{S}$ as the energy flux density.

Using the definition for $\underline{S}$ we can express Poynting's theorem more compactly:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V u \, d\tau - \oint_S \underline{S} \cdot d\underline{a}$$

If no work is done on charges in a volume V, e.g. we are in a region of empty space where there is no charge, in that case $dW/dt = 0$

$$\frac{d}{dt}\int_V u \, d\tau = \oint_S \underline{S} \cdot d\underline{a}$$

$$\Rightarrow \int_V \frac{du}{dt} d\tau \underset{\substack{\uparrow \\ \text{applying again} \\ \text{the div theorem}}}{=} -\int (\nabla \cdot \underline{S}) \, d\tau$$

$$\therefore \frac{\partial u}{\partial t} = -\nabla \cdot \underline{S}$$

looks like a continuity equation for energy. Here energy density u plays the role of charge density ρ, and $\underline{S}$ plays the role of $\underline{J}$ (current density).

This is a statement of local conservation of EM energy.

However, in general, EM energy by itself is not conserved. The fields do work on the charges & the charges in turn generate fields, so energy is "tossed" back & forth between them. We need to include the contributions of both matter (i.e. the charges) and the fields.

Thoughts on the Poynting vector

* $\frac{\partial u}{\partial t} = -\nabla \cdot \underline{S} \Rightarrow \frac{\partial U}{\partial t} = -\int_V \nabla \cdot \underline{S} \, d\tau \underset{\substack{\uparrow \\ \text{stokes} \\ \text{or div theorem}}}{=} -\int_A \underline{S} \cdot d\underline{A} = -\underset{S}{\Phi}(A)$

the rate of change of EM energy in volume V is given by the ~~poynting vector~~ flux of the Poynting vector $\underline{S}$ through surface A.

This eq has a minus sign so if dA points outward ⇒ U increases when $\underline{S}$ is opposite to $d\underline{A}$.

* $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ points in the direction of EM energy flow
points in the direction of wave propagation.

* $|\underline{S}| = \frac{1}{\mu_0} |\underline{E} \times \underline{B}| = \frac{EB}{\mu_0}$ since $E \perp B$ so $\sin\theta = 1$

the magnitude of S is known as the intensity.

* Units of S:

$$[S] = \frac{[E][B]}{[\mu_0]} = \frac{\left(\frac{N}{C}\right)(T)}{\frac{N}{A^2}} = \frac{A^2 \cancel{N} T}{C \cancel{N}} = \frac{A^2 T}{C} = \frac{A^2 \, kg \cdot s^{-2} \cdot A^{-1}}{C}$$

$$= \frac{A \cdot kg \cdot s^{-2}}{C} \underset{\text{since } [A] = \frac{C}{s}}{=} kg \cdot s^{-3}$$

and defining $W \equiv$ ~~watt~~ $\equiv$ Joule/second $= kg \cdot m^2 \cdot s^{-3}$

$$[S] = \frac{W}{m^2} \qquad \frac{\text{power}}{\text{area}}$$

makes sense since we said that the flux of S is the power through area A.

EXAMPLE: suppose the electric & magnetic components of a plane EM wave propagating in the +x direction are:

$$\underline{E} = E_0 \cos(kx - \omega t)\,\hat{j}$$
$$\underline{B} = B_0 \cos(kx - \omega t)\,\hat{k}$$

What is the Poynting vector?

~~With~~
Using our definition:

$\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ so in this case:

$$\underline{S} = \frac{1}{\mu_0} \left[E_0 \cos(kx - \omega t)\,\hat{j}\right] \times \left[B_0 \cos(kx - \omega t)\,\hat{k}\right] \underset{\text{since } \hat{j} \times \hat{k} = \hat{i}}{=} \frac{E_0 B_0}{\mu_0} \cos^2(kx - \omega t)\,\hat{i}$$

We can see that as expected $\underline{S}$ points in the direction of wave propagation.

The ~~[illegible]~~ average energy density is given by:

$$I = \langle |\underline{S}| \rangle = \frac{E_0 B_0}{\mu_0} \underbrace{\langle \cos^2(kx - \omega t) \rangle}_{1/2} = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{c B_0^2}{2\mu_0} \quad (*)$$

~~So, the average intensity is $I = \langle |S| \rangle = \frac{cB_0^2}{2\mu_0}$~~

Now to relate the intensity to the energy density we will first note the equality between the electric & magnetic energy densities.

$$u_B = \frac{B^2}{2\mu_0} = \frac{(E/c)^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{\varepsilon_0 E^2}{2} = u_E$$

considering that for EM waves $B = \frac{E}{c}$

Now we can calculate the average total energy density:

$$\langle u \rangle = \langle u_E + u_B \rangle = \left\langle 2\left(\frac{\varepsilon_0 E^2}{2}\right)\right\rangle = \varepsilon_0 \langle E^2 \rangle = \frac{\varepsilon_0}{2} E_0^2 \quad (**)$$

$$= \frac{1}{\mu_0}\langle B^2 \rangle = \frac{B_0^2}{2\mu_0}$$

following the same steps as for E.

• using that $E = E_0 \cos(kx - \omega t)$ and $\langle \cos^2(kx-\omega t)\rangle = \frac{1}{2}$

So now if we compare this expression for the average energy density $(**)$ with the expression for the intensity $*$, we can see that:

$$I = \langle S \rangle = c \langle u \rangle$$

EXAMPLE: Solar constant.

The magnitude of the Poynting vector at the upper surface of the Earth's atmosphere is referred to as the ~~[illegible]~~ solar constant.

solar constant $\langle S \rangle = 1.35 \times 10^3 \, W/m^2$

a) Assuming that the sun's EM radiation is a plane sinusoidal wave, what are the magnitudes of E & B?

b) What is the total time averaged power radiated by the sun? Consider the mean sun-earth distance is $R = 1.5 \times 10^{11}$ m

We just derived by that for planar EM waves:

$$\langle S \rangle = c \langle u \rangle = c\left(\frac{\varepsilon_0}{2} E_0^2\right) \Rightarrow E_0 = \left[\frac{2\langle S \rangle}{c\varepsilon_0}\right]^{1/2}$$

So we can calculate the magnitude of E ~~and then we~~ if we know $\langle S \rangle$, after that we can use the relation $B_0 = \frac{E_0}{c}$ to obtain B.

From the ~~old~~ problem statement we already know the value of $\langle S \rangle$ so we just need to substitute it in the above eq:

$$E_0 = \left[\frac{2\langle S \rangle}{c\varepsilon_0}\right]^{1/2} = \left[\frac{2(1.35 \times 10^3 \, W/m^2)}{(3\times 10^8 \, m/s)(8.85 \times 10^{-12} C^2/N \cdot m^2)}\right]^{1/2}$$

$$= 1.01 \times 10^3 \, \frac{V}{m}$$

And the corresponding amplitude of the magnetic field is:

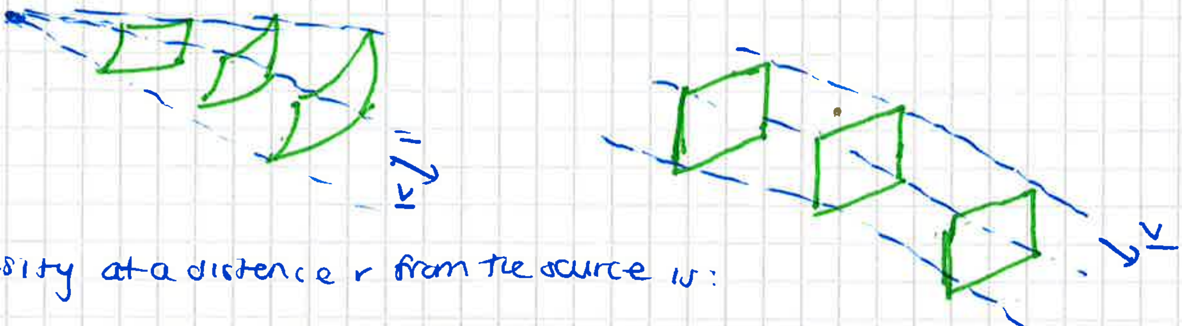
$$B_0 = \frac{E_0}{c} = \frac{1.01 \times 10^3 \, V/m}{3 \times 10^8 \, m/s} = 3.4 \times 10^{-6} \, T$$

b) The total time averaged power radiated by the sun a distance R is:

$$\langle P \rangle = \langle S \rangle A = \langle S \rangle (4\pi R^2) = (1.35 \times 10^3\ W/m^2)(4\pi)(1.5 \times 10^{11}\ m)^2$$
$$= 3.8 \times 10^{26}\ W$$

In this case we have a spherical wave, which originates from a "point like" source.

spherical wave   plane wave



The intensity at a distance r from the source is:

$$I = \langle S \rangle = \frac{\langle P \rangle}{4\pi R^2}$$

In this case the intensity decreases as $\frac{1}{r^2}$, while you can see that for a plane wave the intensity ~~[illegible]~~ remains constant & there is no spreading in energy.

~~[illegible]~~
Momentum & radiation pressure

Since EM waves carry energy, it is not surprising that they can carry energy as well. Since the waves carry momentum, they can also exert a radiation pressure on a surface due to the absorption and reflection of the momentum.

If a plane EM wave is completely absorbed by ~~the~~ a surface, the momentum transfered is related to the energy absorbed by:

$$\boxed{\Delta p = \frac{\Delta U}{c}}$$ complete absorption

($p \equiv$ momentum, $U \equiv$ energy, $c \equiv$ speed of light)

On the other hand if an EM wave is completely reflected by a surface (e.g. a mirror) the result becomes:

$$\boxed{\Delta p = \frac{2\Delta U}{c}}$$ complete reflection

For the complete absorption case, the average radiation pressure (force/unit area) is given by:

$$P = \frac{\langle F \rangle}{A} = \frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle = \frac{1}{Ac}\left\langle \frac{dU}{dt} \right\rangle$$

($P \equiv$ radiation pressure, $F \equiv$ force, $A \equiv$ area)

And since the rate of energy delivered to the surface is:

$$\left\langle \frac{dU}{dt} \right\rangle = \langle S \rangle A = IA$$

(where $I \equiv$ intensity, $A \equiv$ area)

We can write the radiation pressure as:

$$\boxed{P = \frac{I}{c}}$$ complete absorption

(Remember $P \equiv$ radiation pressure
$I \equiv$ intensity
$c \equiv$ speed of light)

Similarly if the radiation is completely reflected, the radiation pressure is:

$$\boxed{P = \frac{2I}{c}}$$ complete reflection

Demo 696 Radiation pressure
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More on plane EM waves. How do we actually "make" one?
We have discussed that EM waves propagate in empty space at the speed of light.
How would we actually generate an EM wave?

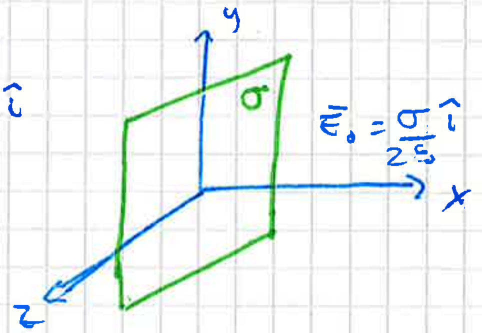
We will discuss this in a simple planar geometry. This is not particularly applicable in the real world but helps us understand how EM waves can be generated, why it takes work to make them & how much energy they carry with them.

How do we make an EM wave?
If we think of a wave on a string, we simply shake the string to make a wave.
↳ we have to do work against the tension in the string when we shake it.
↳ this work we put into the system is carried off as energy flux in the wave.

Can we make an analogy with EM waves?
We can think of $\underline{E}$ as the "string"
Then there is a tension associated w/ the electric field line → there is a restoring force that resists the "shaking"
Then a wave will propagate along the field line as a result of the shake.

To understand what happens we'll use everything we've learned so far + the "reasonable" assumption that EM information propagates at the speed of light c in vacuum.

Suppose we have an ∞ sheet of charge located in the yz-plane, initially at rest with surface charge density σ:



$E_0 = \frac{-\sigma}{2\varepsilon_0}\hat{\imath}$ $\quad$ $E_0 = \frac{\sigma}{2\varepsilon_0}\hat{\imath}$

sheet of charge σ in the yz plane
~~also~~ surface charge density σ

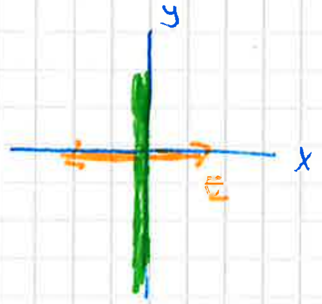
If the sheet is at rest, this surface will give rise to the static $\underline{E}$ field:

$$\underline{E}_0 = \begin{cases} \sigma/2\varepsilon_0 \,\hat{\imath} & x > 0 \\ -\sigma/2\varepsilon_0 \,\hat{\imath} & x < 0 \end{cases}$$

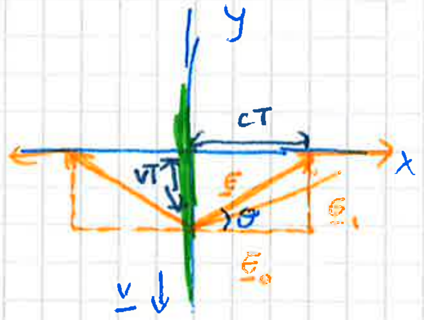
(we've derived this before using Gauss's law)

Now at $t=0$ we start "pulling" on the sheet downward with constant velocity:
$\underline{v} = -v\hat{\jmath}$

What will happen at a later time $t = T$?



at $t < 0$
$\underline{v} = 0$
$\underline{E}_0 = \pm \frac{\sigma}{2\varepsilon_0}\hat{i}$



at $t = T$
$\underline{v} = -v\hat{j}$

If we look at ~~the~~ what happens to $\underline{E}$ generated by the charge on a fixed point in the sheet we will see that as the sheet moves downward the point at $y = 0$ ~~will now be at~~ at $t = 0$

will be at $y = -vT$ at $t = T$ ~~...~~

we have assumed that ~~...~~ that this ~~...~~ if information propagates at the speed of light c, the portion of the field line located at $x > cT$ along the x axis "doesn't know" the charges are moving.

From the schematic you can see that the electric field will now be $\underline{E} = \underline{E}_0 + \underline{E}_1$ for $0 < |x| < cT$

And from the schematic we can also see that:

$$\tan\theta = \frac{E_1}{E_0} = \frac{vT}{cT} = \frac{v}{c} \qquad \text{where } E_1 = |\underline{E}_1| ;\ E_0 = |\underline{E}_0|$$

θ is the ∠ w/ the x-axis.

Now if we remember that $E_0 = \sigma/2\varepsilon_0$, we can write E_1 in terms of E_0 as:

$$\underline{E}_1 = \left(\frac{v}{c}E_0\right)\hat{j} = \left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{j}$$

; $\underline{E}_1$ is the "perturbation" to the static electric field due to the movement of the sheet.

You can see that the direction of $\underline{E}_1$ is such that the forces it exerts on the charges in the sheet resist the motion of the sheet.

i.e there is an upward $\underline{F}$ on the sheet when we try to move it downward. So this is the "tension" we were talking about.

For an infinitesimal area dA of the sheet containing charge $dq = \sigma dA$ the upward tension associated w/ the electric field is:

$$d\underline{F}_e = dq\,\underline{E}_1 = (\sigma dA)\left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{j} = \left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{j}$$

so the external $\underline{F}_{ext}$ that needs to be applied to overcome this tension is:

$$d\underline{F}_{ext} = -d\underline{F}_e = -\left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{j}$$

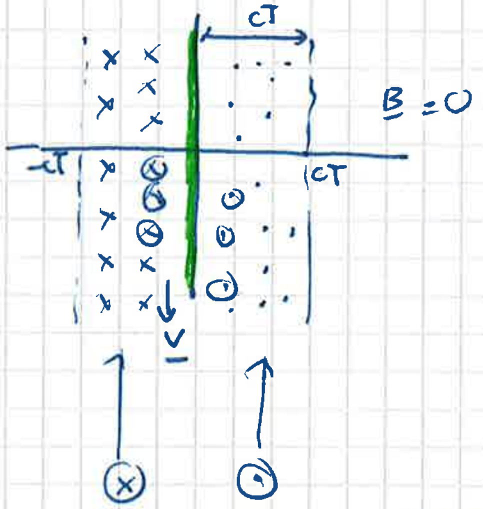
And remembering ~~also~~ that $dW_{ext} = \underline{F}_{ext}\cdot d\underline{s}$ ~~is the work done/unit time per unit~~

the work done per unit time per unit area by the external agent is:

$$\frac{d^2 W_{ext}}{dA\,dt} = \frac{d\underline{F}_{ext}\cdot d\underline{s}}{dA\,dt} = \frac{d\underline{F}_{ext}}{dA}\cdot\frac{d\underline{s}}{dt} = \left(\frac{-v\sigma^2}{2\varepsilon_0 c}\hat{j}\right)\cdot\left(-v\hat{j}\right) = \frac{v^2\sigma^2}{2\varepsilon_0 c}$$

In the process of moving the sheet we also created some current, $\underline{K} = -\sigma v\hat{j}$
And from Ampère's law we know that this will create a magnetic field:

$$\underline{B}_1 = \begin{cases} (\mu_0\sigma v/2)\hat{k} & x>0 \\ -(\mu_0\sigma v/2)\hat{k} & x<0 \end{cases}$$



again the perturbation will not be felt for $x > cT$ & $x < -cT$ so in those regions $\underline{B} = 0$ but there will be a magnetic field for the part in between.

indicate $\underline{B}$ points in opposite directions in opposite sides of the sheet.

The magnetic field $\underline{B}_1$ generated by the current is $\perp$ to $\underline{E}_1$ & has a magnitude $B_1 = \frac{E_1}{c}$ as expected for a transverse EM wave.

What is the energy "carried away" by these perturbation fields B_1 & E_1?
The energy flux is given by the Poynting vector as discussed. For $x>0$:

$$\underline{S} = \frac{1}{\mu_0}\underline{E}_1\times\underline{B}_1 = \frac{1}{\mu_0}\left(\frac{v\sigma}{2\varepsilon_0 c}\hat{j}\right)\times\left(\frac{\mu_0\sigma v}{2}\hat{k}\right) = \left(\frac{v^2\sigma^2}{4\varepsilon_0 c}\right)\hat{i}$$

(half of the work we applied to the sheet)

The fields on the left of the sheet carry the same amount of energy flux to the left.

$\therefore$ the total energy flux carried off by the perturbation $\underline{E}_1$ & $\underline{B}_1$ fields is exactly equal to the rate of work / unit area to pull the charged sheet down against the tension in $\underline{E}$.

From this you can see how you could generate a sinusoidal wave with angular frequency ω by "pulling" the sheet up & down at vel $\underline{v}(t) = -v_0\cos\omega t\,\hat{j}$, this oscillating charge will give fields:

$$\left.\begin{aligned} \underline{E}_1 &= \frac{c\mu_0\sigma v_0}{2}\cos\omega\left(t-\frac{x}{c}\right)\hat{j} \\ \underline{B}_1 &= \frac{\mu_0\sigma v_0}{2}\cos\omega\left(t-\frac{x}{c}\right)\hat{k} \end{aligned}\right\}\text{ for } x>0 \qquad \left.\begin{aligned} \underline{E}_1 &= \frac{c\mu_0\sigma v_0}{2}\cos\omega\left(t+\frac{x}{c}\right)\hat{j} \\ \underline{B}_1 &= \frac{-\mu_0\sigma v_0}{2}\cos\omega\left(t+\frac{x}{c}\right)\hat{k} \end{aligned}\right\}\ x<0$$