Lecture 19. Inductance and Magnetic energy

Last time:

* Motional emf: Induced on a conductor that moves through a magnetic field:

$$\mathcal{E} = \oint \underline{F}_B \cdot d\underline{s}$$

$$\Leftrightarrow \boxed{\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}}$$ where $\Phi_B \equiv \int \underline{B} \cdot d\underline{a}$ is the magnetic flux through a loop

* Lenz's law: Gives us the direction of the current induced due to the motional emf.

$\boxed{\text{The induced current opposes a change in flux of B}}$

$\Leftrightarrow$

$\boxed{\text{Nature abhors a change in flux}}$

* Faraday's law:

A changing magnetic field induces an electric field

$\Leftrightarrow$

$$\oint \underline{E} \cdot d\underline{\ell} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a} = -\frac{d\Phi}{dt}$$

$\Leftrightarrow$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

The end of electrostatics

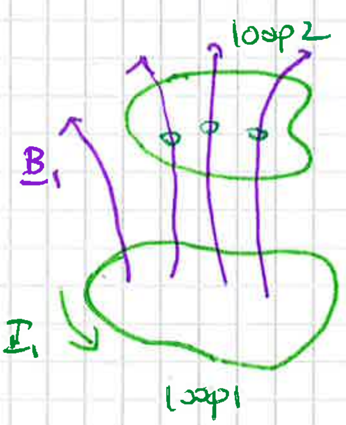
$$\oint \underline{E} \cdot d\underline{\ell} \neq 0$$

$\Leftrightarrow$

$$\nabla \times \underline{E} \neq 0$$

Today: Inductance and magnetic energy

A change in Φ_B will give rise to an emf. Let's assume we have 2 loops in proximity of each other. What is the emf induced in one loop due to a change in current in the other?

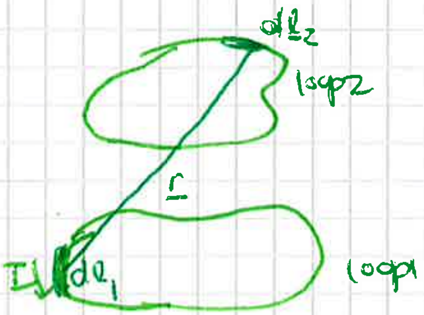


* If we run a steady current around loop 1, it will produce a magnetic field $\underline{B}_1$

* some of the field lines will pass through loop 2 $\rightarrow$ change in flux of $\underline{B}$ $\rightarrow$ induced emf $\rightarrow$ induced current.

How much current will be induced?

Let's look again at the loops:



To know the current induced we would need to know $\underline{B}_1$. Actually calculating $\underline{B}_1$ can be complicated, but looking at Biot-Savart's law we can see that $\underline{B}_1 \propto I_1$:

$$\underline{B}_1 = \frac{\mu_0}{4\pi} I_1 \int \frac{d\underline{\ell}_1 \times \hat{r}}{r^2}$$

If $\underline{B}_1$ is proportional to I_1, this means that the flux through loop 2 is also proportional!

$$\Phi_2 = \int \underline{B}_1 \cdot d\underline{a}_2 = \underbrace{M_{21}}_{\text{constant of proportionality}} I_1 \quad *$$

constant of proportionality $\equiv$ Mutual inductance

statement that Φ_2 is proportional to I_1

We can derive a formula for M_{21} by expressing Φ_2 in terms of the vector potential and using Stokes theorem:

$$\Phi_2 = \int \underline{B}_1 \cdot d\underline{a}_2 \underset{\text{by definition of } \underline{A}_1}{=} \int (\nabla \times \underline{A}_1) \cdot d\underline{a}_2 \underset{\text{stokes theorem}}{=} \oint \underline{A}_1 \cdot d\underline{l}_2 \quad **$$

And we saw in Lecture 15 that for a loop of current A_1 is:

$$\underline{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\underline{l}_1}{r}$$ so subs this in our expression for the flux $**$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\frac{d\underline{l}_1}{r}\right) \cdot d\underline{l}_2 \underset{\text{from eq }*}{=} M_{21} I_1 \quad ***$$

$$\therefore \boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\underline{l}_1 \cdot d\underline{l}_2}{r}}$$ Neumann formula

Thoughts on the Neumann formula:

*Involves a double line integral. One integration around loop 1, another around loop 2.
*Not very useful for practical calculations. But reveals a couple of important facts.
* M_{21} is purely geometrical, i.e only depends on dl_1 & dl_2 (geometry of loops).
(Remember capacitance? This seems similar).
*If we switch the roles of loop 1 & loop 2 the integral remains unchanged.

$$M_{21} = M_{12} = M$$

The flux through loop 2 when we run a current I around loop 1 is identical to the flux through 1 when we run the same current on 2.

(If you want to see a more detailed proof see Purcell 7.7)

$M \equiv$ mutual inductance

As we vary the current on loop 1, we will get a change in flux through loop 2, that will then result in an emf:

$$\mathcal{E}_2 = \frac{-d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

using eq $***$ to subs Φ_2
we can take M out of the derivative since it only depends on ~~the~~ geometry of loops.

and as we know this emf will induce a current on loop 2 given by ohm's law

$$\mathcal{E}_2 = R_2 I_2$$

Let's pause to think about this for a moment.

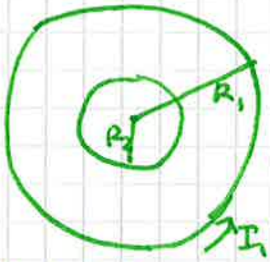
Loop 1 & Loop 2 are NOT connected, yet we can induce a current on loop 2 by changing the current on loop 1.

The units for mutual inductance are:

$[M] = \text{henry}$

$1\ \text{henry} = 1\ H = \dfrac{T\cdot m^2}{A} = \dfrac{V\cdot s}{A}$ where T = tesla, m = meter, A = ampere

Example: What is the mutual inductance of 2 concentric, coplanar loops of radii R_1 & R_2 with $R_1 >> R_2$



We know that the ~~flux~~ mutual inductance is the prop. constant between Φ_2 and I_1

$$\Phi_2 = M I_1$$

We will use this relation to calculate M. So we need to know Φ_2, which means we need to know B_1.

Notice that the converse is also true, we have that $\Phi_1 = M I_2$, but in this case we would need to know B_2 to calculate Φ_1, which is more complicated than B_1. To get a "nice" expression for B_1 we will use the fact that $R_1 >> R_2$.

Let's calculate B_1 then. In lecture 14 we had calculated the expression for the magnetic field of a loop of current at a point in the z axis:

$$B_z(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$, this expression at $z=0$ reduces to:

$$B_0 = \frac{\mu_0 I}{2R}$$

So the magnetic field at the center of the ring due to I_1 in the outer loop is:

$$B_1 = \frac{\mu_0 I_1}{2R_1}$$

Since $R_1 >> R_2$ we approximate the magnetic field through the entire inner loop by B_1. So the flux through this loop is:

$$\Phi_2 = B_1 A_2 = \left(\frac{\mu_0 I}{2R_1}\right)\pi R_2^2 = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

$\Phi = B_1 \int da_2$

and so now we can use the expression $\Phi_2 = M I_1 \Rightarrow M = \dfrac{\Phi_2}{I_1}$

$$\therefore M = \frac{\mu_0 \pi \cancel{I_1} R_2^2}{2R_1 \cancel{I_1}} = \boxed{\frac{\mu_0 \pi R_2^2}{2R_1}}$$

And we can see that as expected M only depends on the geometry of the loops. & is independent of the current through the coil.

~~Demo 465, Establishment of a current in an inductor~~

Self inductance

If you think carefully about it, a changing current will not only induce an emf on nearby loops, it also induces an emf on the source loop itself. Using the same arguments as before we can see that B the field and as a consequence the flux is proportional to the current induced:

$$\Phi = LI \quad \text{where } L \equiv \text{self inductance}$$

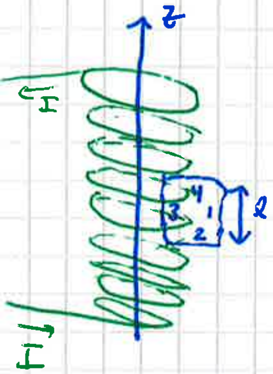
and if the current changes in time, we will have an induced emf on the loop:

$$\mathcal{E} = -L\frac{dI}{dt}$$

Inductance like capacitance is intrinsically positive. From the equation above we see that the minus sign enforces Lenz's law → the emf is in such a direction to oppose any change in current. For this reason we call this **back emf**.

Physically, the inductance L is a measure of the "resistance" to a change in current. Inductance can be thought of as the electrical analogue of mass in mechanical systems: The greater L is, the harder it is to change the current, just as the greater the mass is, the harder it is to change an object's velocity.

EXAMPLE: Compute the self inductance of a solenoid with N turns, length ℓ and radius R with a current I flowing through each turn.



Similar to the problem for mutual inductances we will use the expression:

$$\Phi = LI \text{ to compute } L\text{, so we need } \Phi.$$

To calculate Φ we need to know B for a solenoid.

We can compute B using Ampère's law

$$\oint \underline{B}\cdot d\underline{s} = \underbrace{\int_1 \underline{B}\cdot d\underline{s}}_{=0} + \underbrace{\int_2 \underline{B}\cdot d\underline{s}}_{=0} + \underbrace{\int_3 \underline{B}\cdot d\underline{s}}_{B\ell} + \underbrace{\int_4 \underline{B}\cdot d\underline{s}}_{=0}$$

$$= B\ell$$

For regions 4 & 2 $\underline{B} \perp$ to $d\underline{s}$ so $\underline{B}\cdot d\underline{s} = 0$
since $\underline{B}$ is non-zero only inside the solenoid we get a contrib only for region 3. Outside due to symmetry the contributions of the loops cancel out. (see direction of I in drawing).

$$\therefore \oint \underline{B}\cdot d\underline{s} = B\ell = \mu_0 I_{enc} = \mu_0 N I \Rightarrow \underline{B} = \mu_0 n I\,\hat{k}$$

where $n = \dfrac{N}{\ell}$

Now that we have the field we can compute the flux through each turn

$$\Phi_B = BA = \mu_0 n I\,(\pi R^2) = \mu_0 n I \pi R^2$$

And because we have N turns: $\Phi_{Tot} = N\Phi_B = (n\ell)(\mu_0 n I \pi R^2) = \mu_0 n^2 \ell I \pi R^2$

Now we can calculate the inductance $\Phi_{tot} = LI \Rightarrow L = \dfrac{\Phi_{tot}}{I} = \boxed{\mu_0 n^2 \ell \pi R^2}$ depends only on geometry

Energy stored in a magnetic field

Demo 4.6.5. Establishment of a current in an inductor

It takes a certain amount of energy to start current flowing in a circuit since an inductor in a circuit opposes any change in current through it.
$\Leftrightarrow$
we must do work to overcome the back emf ~~to~~ and get the current going.

This is a fixed amount of work & it is recoverable: you get it back when the circuit is turned off.

We can think of this as energy "latent" in the circuit, we will see that this is indeed energy stored in the magnetic field. The role played by an inductor in the magnetic case is analogous to a capacitor in the electric case.

The work done per unit charge against the back emf, in one trip around the circuit is ~~precisely~~ $-\mathcal{E}$. We add a - sign because it's work done by you, not by the emf. The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}I \underset{\substack{\uparrow \\ \text{by definition of } \mathcal{E}}}{=} -\left(-L\frac{dI}{dt}\right)I = LI\frac{dI}{dt} \quad *$$

~~Thus as~~ From this eq we can see that

if $\frac{dI}{dt} > 0$ (current increasing) then the external source is doing positive work to transfer energy to the inductor

$\frac{dI}{dt} < 0$ current decreasing. External source takes energy away from the inductor.

We can obtain the total work done by the external source to increase the current from zero to I by integrating eq $*$ with respect to time:

$$\int dW = \int_0^I LI'dI' \Rightarrow W = \frac{1}{2}LI^2$$

which is equal to the magnetic energy stored in the inductor:

$$\boxed{U_B = \frac{1}{2}LI^2}$$

This expression is analogous to the electric energy stored in a capacitor
$$U_E = \frac{1}{2}\frac{Q^2}{C}$$

Note: Energy is stored not dissipated in an inductor. Energy flows into an inductor when $\frac{dI}{dt} > 0$, then it is released later when the current decreases $\frac{dI}{dt} < 0$.

This is different that the case of a resistor where energy will flow into it and be dissipated in the form of heat, regardless if I is steady or changing in time.

EXAMPLE: Energy stored in a solenoid

A long solenoid ~~consists~~ with length ℓ and radius R consists of N turns of wire. A current passes through the coil. Find the energy stored in the system.

We just derived the expression for energy:

$$U_B = \frac{1}{2} L I^2$$

earlier in the lecture we derived the self inductance for a solenoid.

$$L = \mu_0 n^2 \ell \pi R^2$$

so with this we get directly:

$$U_B = \frac{1}{2}\left(\mu_0 n^2 \ell \pi R^2\right) I^2 = \frac{1}{2}\mu_0 n^2 I^2 \pi R^2 \ell$$

For a solenoid we have that $B = \mu_0 n I$ so we can write U_B in terms of B:

$$U_B = \frac{1}{2\mu_0}(\mu_0 n I)^2 (\pi R^2 \ell) = \frac{B^2}{2\mu_0}\underbrace{(\pi R^2 \ell)}_{\text{volume of the solenoid}}$$

so we can write an expression for the magnetic energy density:

$$u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0}$$ energy per unit volume of the magnetic field.

in general even for non uniform B:

$$\boxed{u_B = \frac{B^2}{2\mu_0}}$$ magnetic energy density

which looks similar to the energy density assoc. with the electric field:

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

RL Circuits

The addition of a time changing magnetic field to a circuit implies that $\oint \underline{E}\cdot d\underline{s} \neq 0$ around the circle. Instead we have that:

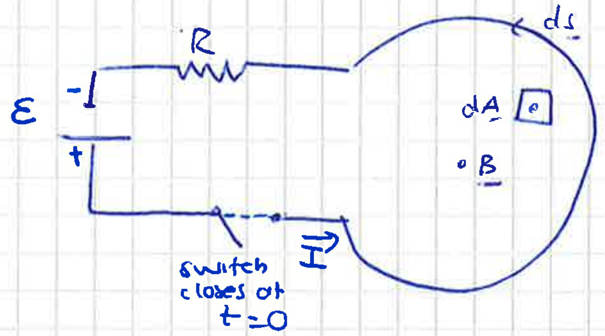
$$\oint \underline{E}\cdot d\underline{s} = -\frac{d}{dt}\int \underline{B}\cdot d\underline{a}$$ Faraday's law

Any circuit where current changes with time (non-steady current) will have time changing magnetic fields & induced currents. How do we solve simple circuits taking those effects into account?

Note that once $\oint \underline{E}\cdot d\underline{s} \neq 0 \Leftrightarrow \nabla\times\underline{E} \neq 0$ we can no longer define a potential, so the electric potential difference between 2 points in a circuit is no longer well defined.

$\Rightarrow$ We can't use electric potential differences to understand circuits, but we can still write in a straight forward way what determines the behavior of ~~a~~ a circuit.

Consider the following circuit:



we have a battery ($\mathcal{E}$)
a resistor (R)
a switch (S)
a "one loop inductor"
$\underline{B}$ pointing out of the page

what are the consequences of having this inductor?

At $t=0$ we close the switch
For $t>0$ current I will flow in the direction shown (from + to - as usual)

What is $I(t)$ for $t>0$?

We need to first know what is $\int \underline{E}\cdot d\underline{s}$ around this circuit. Let's look at the contributions to this integral around the circle.

we have chosen the direction of $d\underline{s}$ to be counterclockwise if we assume B is out of the page.

1) First contribution to the integral is the battery $-\mathcal{E}$, it has a $-$ sign since $\underline{E}$ is directed from the positive to the negative terminal, which is opposite to the direction of $d\underline{s}$.

2) Then there is an electric field in the resistor in the direction of the current so when we move through the resistor in that direction $\underline{E}\cdot d\underline{s} > 0$, so the contrib to the line integral is $+IR$

3) What about when we move through the loop inductor? There is no electric field, if the resistance of the wire making up the loop is zero.

$\therefore$ going around the loop clockwise against the current:

$$\oint \underline{E}\cdot d\underline{s} = -\mathcal{E} + IR$$

Now, what is the magnetic flux Φ_B through the surface ~~of the loop~~ of the circuit? We'll consider that the area encompassed by the resistor ~~+~~, battery is negligible compared to the loop inductor.

$\Phi_B > 0$ in that part since B points out of the page, which is the same direction we assume for $d\underline{a}$. $\Rightarrow \underline{B}\cdot d\underline{a} > 0$

From our definition of (self) inductance:

$$\Phi = LI \Rightarrow \frac{d\Phi}{dt} = L\frac{dI}{dt} \quad *$$

Now we will apply Faraday's law: $\oint \underline{E}\cdot d\underline{l} = -\frac{d\Phi}{dt} \quad **$

so combining * and ** :

$$\oint \underline{E}\cdot d\underline{l} = -L\frac{dI}{dt} = -\mathcal{E} + IR$$

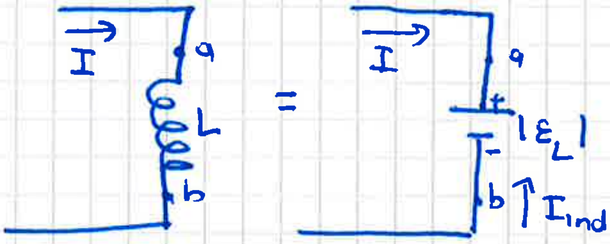
eq

We can write this eq as:

$$\Delta V = \mathcal{E} - IR - L\frac{dI}{dt} = 0$$

∴ This last expression has been cast in a form that resembles Kirchhoff's loop rule, namely : Σ of potential drops around a loop is zero. To preserve the loop rule we must specify the "potential drop" across an inductor.

For $\frac{dI}{dt} > 0$ (increasing current)



$$\varepsilon_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) < 0$$

For $\frac{dI}{dt} < 0$ (decreasing current)

$$\varepsilon_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) > 0$$

According to Lenz's law: The polarity of the self induced current is such to oppose the change in current.

If the current is increasing, I_{ind} will be opposite in direction to I.
If the current is decreasing, I_{ind} will be in the same direction as I.

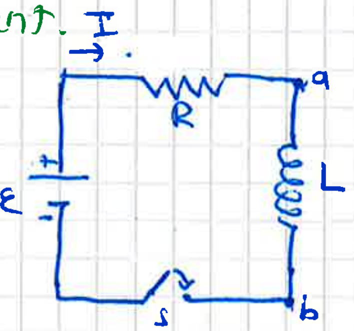
In both cases $\left(\frac{dI}{dt} > 0 \text{ \& } \frac{dI}{dt} < 0\right)$ the "change in potential" when moving from a to b along the direction of I is:

$$V_b - V_a = -L\left(\frac{dI}{dt}\right)$$, so we have:

Kirchhoff's Loop rule modified for inductors
If an inductor is traversed in the direction of the current the "potential change" is $-L\left(\frac{dI}{dt}\right)$. On the other hand, if the inductor is traversed in the direction opposite to the current, the "potential change" is $+L\left(\frac{dI}{dt}\right)$

Using this modified rule we will get the correct equations for circuits that contain inductors. But this is wrong, the arguments we made have no physical basis, we can't define a potential drop across an inductor. Kirchhoff's loop rule was based on the fact that $\oint \vec{E_s} \cdot d\vec{l} = 0$, which is not the case at all. In fact it is $+L\left(\frac{dI}{dt}\right)$. Let's look at these cases in detail!

Rising current. $\frac{dI}{dt} > 0$



Consider the circuit drawn on the left. At time $t = 0$ the switch is closed. Current doesn't rise immediately to its max value $\frac{\varepsilon}{R}$, because we have the inductor (L), which has a self induced emf

Using the modified Kirchhoff's law & considering it's a rising current: $\left(\frac{dI}{dt} > 0\right)$

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \quad (*)$$

Note: The potential difference across R depends on I
The potential difference across L depends on $\frac{dI}{dt}$ change in current
The induced current does not oppose I, but its change.

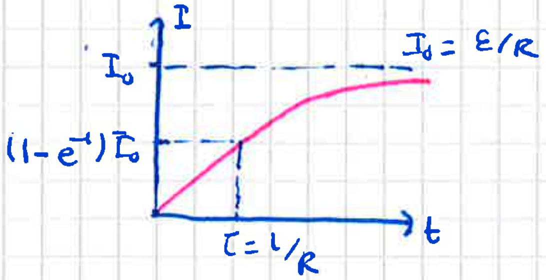
$$\Rightarrow \frac{dI}{I - \mathcal{E}/R} = -\frac{dt}{L/R}$$

Integrating both sides and using the initial condition that at $t=0$ $I=0$

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

where $\tau = \frac{L}{R}$ is the time constant of the RL circuit.

The qualitative behavior of the current as a function of time is



After some time I reaches its eq value $\frac{\mathcal{E}}{R}$
τ measures how fast eq is attained
The larger the value of L, the longer it takes to build up current.

Similarly we can obtain the magnitude of the induced emf:

$$|\mathcal{E}_L| = \left|-L\frac{dI}{dt}\right| = \mathcal{E}e^{-t/\tau}$$

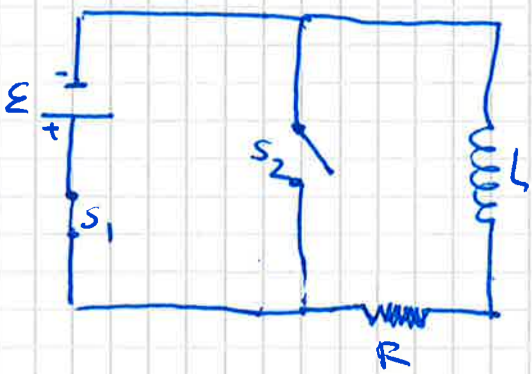
, you can see its max at $t=0$ and vanishes as $t \to \infty$
$\Rightarrow$ after sufficiently large t, L behaves like a conducting wire.

To see that energy is conserved we can multiply eq $(*)$ by I:

$$\mathcal{E}I - I^2R - IL\frac{dI}{dt} = 0 \Rightarrow \underbrace{I\mathcal{E}}_{\text{rate at which the battery delivers energy to the circuit}} = \underbrace{I^2R}_{\text{power dissipated by resistor in the form of heat}} + \underbrace{LI\frac{dI}{dt}}_{\text{Rate at which energy is stored in the inductor}}$$

Decaying current

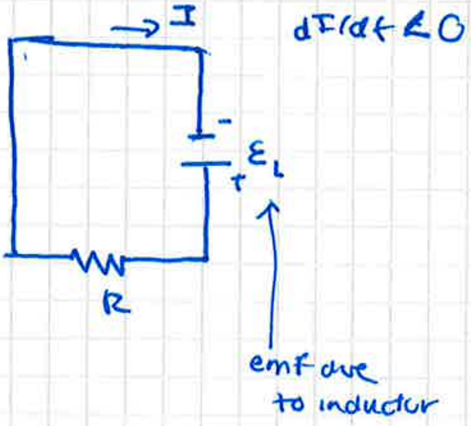
Now we will consider a case where $\frac{dI}{dt} < 0$, i.e current is decreasing. This would happen in the following situation:



As it is drawn, we have the same circuit as before.
If S_1 has been closed for a while current has reached it's steady state value $\frac{\mathcal{E}}{R}$.

What happens to the current when at $t=0$ S_1 is opened and S_2 closed?

It helps to draw the equivalent circuit



If we now apply Kirchhoff's loop rule for the case of decreasing current:

$$|\varepsilon_L| - IR = -L\frac{dI}{dt} - IR = 0$$

$$\Rightarrow \frac{dI}{I} = -\frac{dt}{L/R}$$

$$\Rightarrow I(t) = \frac{\varepsilon}{R}e^{-t/\tau} \quad \text{where } \tau = \frac{L}{R}$$

time constant.

In this case we can see that current decays as: