Final Exam

29.01.2025

The time available for the exam is 3 hours. No calculators, books or scripts are allowed, only one doublesided A4 handwritten paper with notes.
Use the space provided at each question and clearly mark your final answer.
Scrap paper is available at the end of the exam sheet.
SI units are implied throughout the exam.

NAME: S. Olution

N°SCIPER: 123456

$$\text{cartesian} \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\text{cylindrical} \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$$

$$\text{spherical} \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

$$\nabla U(r,\phi,z) = \begin{bmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial U}{\partial \phi} \\ \frac{\partial U}{\partial z} \end{bmatrix} \qquad \nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \qquad \nabla^2 \vec{A} = \begin{bmatrix} \nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} \\ \nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} \\ \nabla^2 A_z \end{bmatrix}$$

$$(\vec{A}\cdot\nabla)\vec{A} = \begin{bmatrix} A_r\frac{\partial A_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi^2}{r} + A_z\frac{\partial A_r}{\partial z} \\ A_r\frac{\partial A_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{A_\phi A_r}{r} + A_z\frac{\partial A_\phi}{\partial z} \\ A_r\frac{\partial A_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial A_z}{\partial \phi} + A_z\frac{\partial A_z}{\partial z} \end{bmatrix}$$

$$\nabla \times \vec{A} = \begin{bmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \\ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \end{bmatrix}$$

Problem 1

Consider a long, hollow cylinder of radius R_1 coaxial with a larger hollow cylinder of radius R_2. The inner cylinder is moving with a velocity v_0 along the direction of its axis ($\hat{z}$), whereas the outer cylinder is fixed. A viscous (η) and incompressible fluid with density ρ occupies only the space inside the inner cylinder ($r < R_1$). A (negative) pressure gradient $-\frac{dP}{dz}$ is applied.

The flow can be considered fully established and for $\mathcal{R}e \leq 10$ it can be considered laminar, gravitational forces can be neglected, and the no-slip condition can be assumed to apply.

a) Determine the velocity field of the flow for $\mathcal{R}e \leq 10$ for $r < R_1$. (3 points)

Poiseuille with different boundary conditions

$V_z(r)$ only $\Rightarrow \frac{dP}{dz} = \eta \nabla^2 \vec{v} = \eta \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = \eta\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right)$

$V_z(r) = \frac{r^2}{4\eta}\frac{dP}{dz} + C_1 \ln(r) + C_2 \qquad r = 0$ finite $\Rightarrow C_1 = 0$

$r = R_1 \rightarrow V_z = V_0 \rightarrow V_0 = \frac{R_1^2}{4\eta}\frac{dP}{dz} + C_2 \rightarrow C_2 = \frac{-R_1^2}{4\eta}\frac{dP}{dz} + V_0$

$V_z(r) = \frac{1}{4\eta}\frac{dP}{dz}(r^2 - R_1^2) + V_0$

b) Determine the total mass of the fluid that passes through the cylinder at a fixed z value in a time interval Δt (2 points)

$Q_{vol} = \int_0^{R_1} v_z(r)\, 2\pi r\, dr = \frac{\pi}{2\eta}\frac{dP}{dz}\int_0^{R_1}(r^3 - R_1^2 r)\,dr + \int_0^{R_1} 2\pi V_0\, r\, dr$

$= -\frac{\pi}{8\eta}\frac{dP}{dz}R_1^4 + \pi V_0 R_1^2$

Total Mass $= Q_{vol}\cdot\rho\cdot\Delta t = \pi\rho\Delta t R_1^2\left(\frac{R_1^2}{8\eta}\frac{dP}{dz} + V_0\right)$

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c) As shown in the lecture, the Reynolds number for flow through a pipe is $\rho \overline{v} D/\eta$ with D the diameter of the pipe and $\overline{v}$ the average velocity. What is the maximum pressure gradient that can be applied for the system described above before the flow becomes turbulent ? (3 points)

$$Re = \frac{\rho \bar{v}\, 2R_1}{\eta}$$ only v relative to pipe relevant

$$\bar{v} = \frac{Q_{vol}}{\pi R_1^2} = \frac{\pi R_1^4}{\pi 8 \eta R_1^2}\left|\frac{dp}{dz}\right| = \frac{R_1^2}{8\eta}\left|\frac{dp}{dz}\right|$$

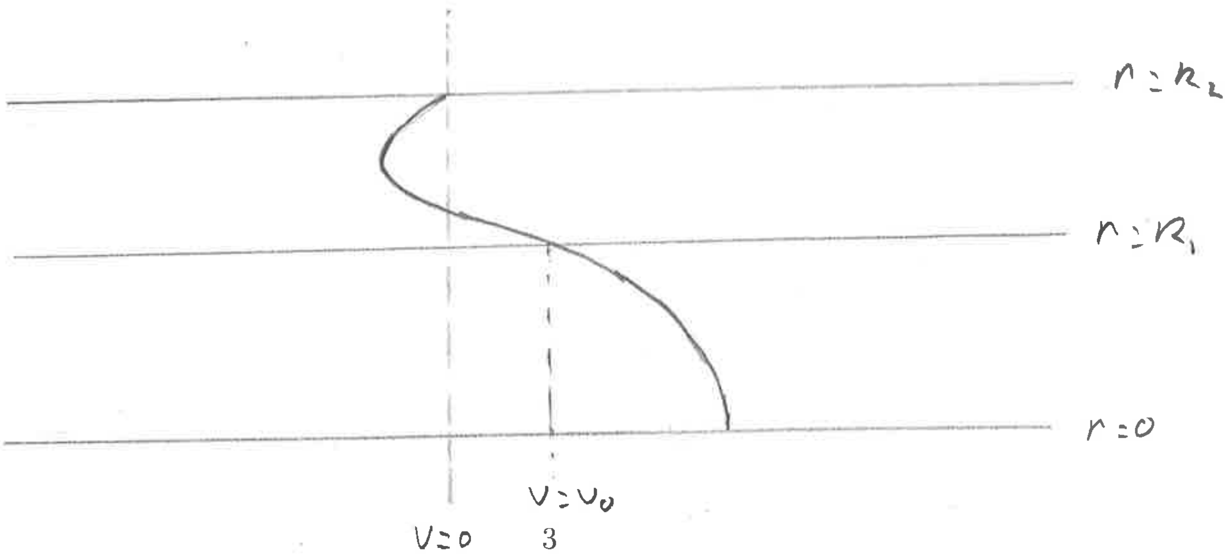
$$Re = \frac{\rho\, 2R_1^3}{8\eta^2}\left|\frac{dp}{dz}\right| = \frac{\rho R_1^3}{4\eta^2}\left|\frac{dp}{dz}\right| < 10$$

$$\left|\frac{dp}{dz}\right| < \frac{40\,\eta^2}{\rho R_1^3}$$

d) In the outer cylindrical shell ($R_1 < r < R_2$) the fluid is flowing back due to an opposite pressure gradient. Sketch the velocity profile $v(r)$ for $0 \leq r \leq R_2$. Assume laminar flow and no further calculation expected. (2 points)

Important : * v_0 at R_1 & $v=0$ at R_2

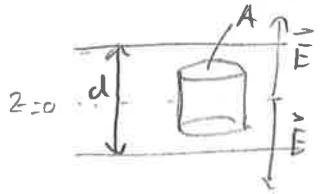
* nett backflow for $R_2 > r > R_1$



Problem 2

Consider an infinite plane slab of thickness d placed along the xy-plane. The slab has volume charge density $\rho = a|z|$ with a a positive constant.

a) Determine the electric field as a function of z where $z = 0$ coincides with the centre of the slab. (3 points). Symmetry: $\vec{E} = E_z \hat{z}$



Gauss: $2AE_z = A \cdot \frac{a}{\varepsilon_0} |z| z \implies \vec{E} = \frac{a}{2\varepsilon_0} |z| z \hat{z}$

for $|z| < d/2$

$|z| \geq d/2$ $\quad 2AE_z = A \cdot 2 \cdot \frac{a}{2\varepsilon_0} \left(\frac{d}{2}\right)^2 \implies \vec{E} = \pm \frac{ad^2}{8\varepsilon_0} \hat{z}$

\+ for $z > d/2$

\- for $z < -d/2$

$Q_{encl} = 2\int_0^z a|z| dz = a|z|z$

$= 2\int_0^{d/2} a|z| dz = 2 \cdot \frac{1}{2} a \left(\frac{d}{2}\right)^2 = \frac{ad^2}{4}$

b) Determine the electrostatic potential as a function of z taking $\Phi = 0$ for $z = 0$. (2 points)

$|z| < d/2$ $\quad \Phi = -\int_0^z \frac{a}{2\varepsilon_0} |z| z \, dz = -\frac{a|z|z^2}{6\varepsilon_0}$

$|z| > d/2$ $\quad \Phi = -\int_0^{d/2} \frac{a}{2\varepsilon_0} |z| z \, dz - \int_{d/2}^z \frac{ad^2}{8\varepsilon_0} dz = -\frac{ad^3}{48\varepsilon_0} - \frac{ad^2 z}{8\varepsilon_0} + \frac{ad^3}{16\varepsilon_0}$

$\Phi = \frac{ad^2}{4\varepsilon_0}\left(\frac{5}{12} d - z\right)$

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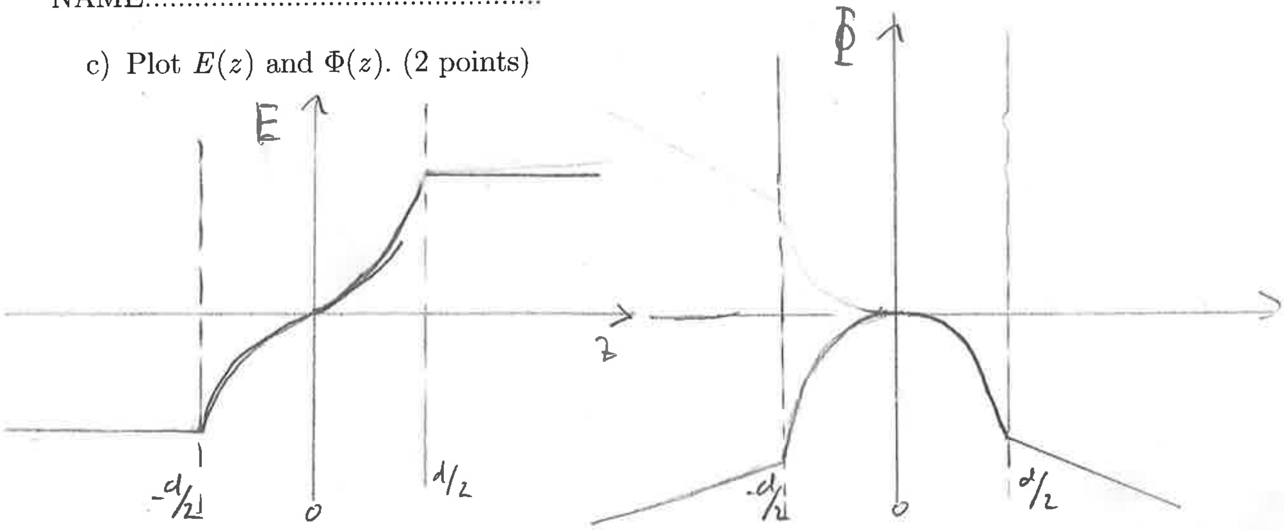
c) Plot $E(z)$ and $\Phi(z)$. (2 points)

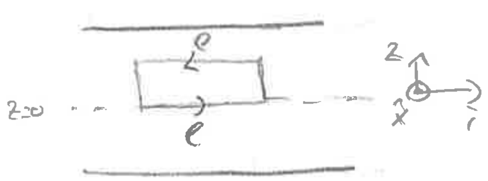


The same slab is now set in motion and moves with constant velocity v_0 along the $\hat{x}$ direction.

d) Determine the direction of the magnetic field $\vec{B}$ for $z > 0$ from the symmetry of the system. (1 point)

Current along $+\hat{x}$, right hand rule: $\vec{B}$ along $-\hat{y}$ for $z > 0$

e) Calculate $\vec{B}(z)$. (3 points)



at $z=0$ $\quad B=0$

$|z| < d/2$

$\oint \vec{B} \cdot d\vec{l} = I_{enc} \qquad Bl = \mu_0 \frac{a}{2} |z| v_0 z l$

$B = \mu_0 \frac{a}{2} |z| z v_0 \Rightarrow \vec{B} = -\mu_0 \frac{a}{2} |z| z v_0 \hat{y}$

$|z| > d/2$: $I_{enc} = a \frac{d^2}{8} v_0 l \rightarrow B = \mu_0 a \frac{d^2}{8} v_0$

$\vec{B} = -\mu_0 a \frac{d^2}{8} v_0 \hat{y} \qquad z > d/2$

$\vec{B} = \mu_0 a \frac{d^2}{8} v_0 \hat{y} \qquad z < -d/2$

f) Find an expression for the vector potential $\vec{A}$. (3 points)
(Hint : $\vec{A}$ is typically in the same direction as a current element. If you did not solve the previous parts, you can take a simple expression with $\vec{B} \neq 0$.)

$\vec{A}$ parallel to $\vec{J}$ + only dependent on z

$\rightarrow \vec{A} = A(z)\hat{x}$

$\vec{B} = \nabla \times \vec{A} \longrightarrow \pm \mu_0 \frac{ad^2 V_0}{8} \hat{y} = \frac{\partial A}{\partial z} \hat{y}$ outside slab

$\Longrightarrow \vec{A} = -\frac{\mu_0}{8} a d^2 V_0 |z| \hat{x}$

$|z| < d/2$ $\quad \frac{\partial A}{\partial z} \hat{y} = -\mu_0 \frac{a}{2} |z| z V_0 \hat{y}$

$\Longrightarrow \vec{A} = -\frac{\mu_0}{6} a |z| z^2 V_0 \hat{x}$

(Depending on average response of students 3 points might be given to only outside slab)

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Problem 3

a) Consider a sphere of radius R made of a dielectric material with dielectric constant ϵ_r. Homogeneously distributed throughout the sphere is a (free) charge density ρ. Determine the electrostatic potential Φ at the centre of the sphere (relative to zero at infinity). (4 points)

$$\oiint \vec{D}\cdot d\vec{S} = Q_{enc} \rightarrow D\, 4\pi r^2 = \rho \frac{4}{3}\pi r^3 \rightarrow D = \frac{1}{3}\rho r$$

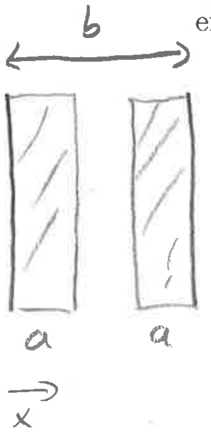
$$\Longrightarrow \vec{E} = \frac{\rho r}{3\varepsilon_0 \varepsilon_r}\hat{r} \qquad r < R$$

$$\vec{E} = \frac{\rho R^3}{3\varepsilon_0 r^2} \qquad r > R \qquad \text{(standard charged sphere)}$$

$$\Phi = -\int_\infty^0 \vec{E}\cdot d\vec{\ell} = -\int_\infty^R \frac{\rho R^3}{3\varepsilon_0 r^2}dr - \int_R^0 \frac{\rho r}{3\varepsilon_0\varepsilon_r}dr = \left[\frac{\rho R^3}{3\varepsilon_0 r}\right]_\infty^R - \left[\frac{\rho r^2}{6\varepsilon_0\varepsilon_r}\right]_R^0$$

$$\Phi = \frac{\rho R^2}{3\varepsilon_0 r} + \frac{\rho}{3\varepsilon_0}\frac{R^2}{2\varepsilon_r} = \frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2\varepsilon_r}\right)$$

b) Consider a parallel plate capacitor with fixed potential difference Φ between the plates. The distance between the plates is b and attached to the plates are two slabs with thickness a of the same dielectric material (ϵ_r), whereby $2a < b$ and there thus is an empty space between the slabs. Determine the D-field in this empty region. (3 points)



$$\int \vec{E}\cdot d\vec{x} = \Phi \Longrightarrow E_r a + E_0(b-2a) + E_r a = \Phi$$

$$\rightarrow \frac{2Da}{\varepsilon_0\varepsilon_r} + \frac{D(b-2a)}{\varepsilon_0} = \Phi$$

$$D\,\frac{2a + \varepsilon_r(b-2a)}{\varepsilon_0\varepsilon_r} = \Phi$$

$$\Longrightarrow D = \frac{\Phi\,\varepsilon_0\varepsilon_r}{2a+\varepsilon_r(b-2a)}$$

Can also be solved by calculating new capacitance → charge on plates → D-field.

Problem 4

In a betatron, a time-varying magnetic field $\vec{B}(t)$ can be used to accelerate particles with charge q and mass m. In the following only velocities $v \ll c$ are considered and $\vec{F} = m\vec{a}$ can be used. Rotational symmetry can be assumed.

a) What is the radius R of the orbit of the particle? (1 point)

$$R = \frac{mv}{qB}$$

b) In an optimal design the radius of the trajectory does not change with time. Use this to express the electric field in terms of R and B. (2 points)
(Hint : differentiate the expression found in (a))

q, m, R constant

$$qBR = mv$$

$$\rightarrow qR\frac{dB}{dt} = m\frac{dv}{dt} = ma = F = qE$$

$$\Rightarrow qE = qR\frac{dB}{dt} \rightarrow E = R\frac{dB}{dt}$$

(Final questions on next page)

c) Determine the ratio in magnitude between $B(R)$ and the average magnetic field in the circle enclosed by R. At $t = 0$ the particle can be assumed to be at rest and $B = 0$. (3 points)

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\varphi}{dt} \longrightarrow E\,2\pi R = -\frac{d\varphi}{dt}$$

$$R\frac{dB}{dt} = -\frac{1}{2\pi R}\frac{d\varphi}{dt} \xRightarrow{\text{integrate}} RB = -\frac{1}{2\pi R}\varphi + \text{const}$$

at $t = 0$ $B = 0$ $\rightarrow$ const $= 0$

$B = -\frac{1}{2}\frac{\varphi}{\pi R^2}$ also $\varphi = \bar{B} \cdot \pi R^2$ $\leftarrow$ average B

$\Rightarrow B(R) = \frac{1}{2}\bar{B}$ (in magnitude)

d) Sketch the possible B-field lines in the plane perpendicular to the orbit. (2 points)