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ON THE SPIN ANGULAR MOMENTUM OF MESONS

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Zusammenfassung

Eine allgemeine Definition für den Spindrehimpuls und die Spinmomentdichte willkürlicher Felder wird gegeben. Angewandt auf den Fall des Meson-Feldes ergibt sich für das Gesamt-Spinmoment genau der Ausdruck von Proca; jedoch trifft die Behauptung von Durandin und Erschow, dass der Zeitmittelwert dieses Momentes bei Quantisierung des Feldes verschwindet, nicht zu.

Resumo

Ĝenerala difino de la spin-movkvanta momanto kaj de la spinmomanta denseco de arbitraj kampoj estas donata. Se tiu difino estas aplikata por la mezona kampo, oni trovas por la entuta spinmomanto ekzakte la esprimon de Proca; tamen la aserto de Durandin kaj Erschow, ke la meznombra valoro de tiu momanto nuliĝas ĉe kvantizado de la kampo, estas malprava.

§ 1. In the fourth paper of Yukawa and collaborators [1]) on the interaction of elementary particles a statement of Durandin and Erschow [2]) has been quoted, that the spin of the quantized field of Proca quanta would have the value 0. As this statement seems to be in contradiction with the fact that three directions of polarization are possible for a Proca quantum with given momentum and charge, I have investigated more in detail how the spin of arbitrary fields generally should be defined and have applied the results to the field of mesons [3]) [4]).

Let the field equations (*first order* differential equations) be derived from a Lagrangian which is the integral over time and space of a Lagrangian function L, scalar with respect to the transformations of special relativity theory. We assume that L does not explicitly contain $ct = x^0 = -x_0$, $x = x^1 = x_1$, $y = x^2 = x_2$,

$z = x^3 = x_3$ and that it depends on these coordinates (x) only as it is a function of the field components *) $q(x)$ and their first derivatives $\nabla_\lambda q(x)$, being linear in the latter. The field equations are given by

$$\frac{\partial L}{\partial q} - \nabla_\lambda \frac{\partial L}{\partial \nabla_\lambda q} = 0, \quad (\nabla_\lambda = \partial/\partial x^\lambda). \tag{1}$$

Let the "orbital" energy-momentum density tensor be defined in the usual way by

$$\mathfrak{t}_{\mu\nu} = Re\,\{t_{\mu\nu}\};\; t_{\mu\nu} = \sum_q \frac{\partial L}{\partial \nabla^\nu q} \nabla_\mu q - g_{\mu\nu} L. \tag{2}$$

It satisfies the continuity equation $\nabla^\nu \mathfrak{t}_{\mu\nu} = 0$. The orbital angular momentum density tensor is then defined by

$$\mathfrak{m}_{\lambda\mu\nu} = x_{[\lambda} \mathfrak{t}_{\mu]\nu} \equiv x_\lambda \mathfrak{t}_{\mu\nu} - x_\mu \mathfrak{t}_{\lambda\nu}. \tag{3}$$

In general $\mathfrak{m}$ will not satisfy a continuity equation, since

$$\nabla^\nu \mathfrak{m}_{\lambda\mu\nu} = \mathfrak{t}_{[\mu\lambda]}\; (\neq 0). \tag{4}$$

Now, by an infinitesimal spatial rotation or a Lorentz transformation of the co-ordinates

$$\delta x^\nu = x_\mu \delta\omega^{\mu\nu},\; \delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}, \tag{5}$$

let the field components q in a fixed point of the space-time manifold be linearly transformed, say by

$$\delta q = \delta\omega^{\mu\nu} S_{\mu\nu\,op}\, q,\; (S_{op} \text{ operating on } q); \tag{6}$$

so that the transformation of their gradients is given by

$$\delta(\nabla_\lambda q) = \nabla_\lambda(\delta q) + (\delta\nabla_\lambda) q = \delta\omega^{\mu\nu} (\nabla_\lambda S_{\mu\nu\,op} - g_{\mu\lambda}\nabla_\nu) q. \tag{7}$$

Putting

$$\mathfrak{f}_{\lambda\mu\nu} = -\mathfrak{f}_{\mu\lambda\nu} = Re\left\{\sum_q \frac{\partial L}{\partial \nabla^\nu q} S_{[\mu\lambda]\,op}\, q\right\};\; \mathfrak{i}_{\lambda\mu\nu} = \mathfrak{m}_{\lambda\mu\nu} + \mathfrak{f}_{\lambda\mu\nu}, \tag{8}$$

we find from the fact that the Lagrangian function $L(q, \nabla q)$ is a scalar field, with the help of (1)—(4) and (6)—(7):

$$\nabla^\nu \mathfrak{i}_{\lambda\mu\nu} = 0. \tag{9}$$

Further we shall put

$$\begin{aligned} &\Theta_{\mu\nu} = \nabla^\lambda f_{\lambda\mu\nu};\quad \mathfrak{s}_{\lambda\mu\nu} = x_{[\lambda}\Theta_{\mu]\nu}; \\ &T_{\mu\nu} = \mathfrak{t}_{\mu\nu} + \Theta_{\mu\nu};\; \mathfrak{j}_{\lambda\mu\nu} = x_{[\lambda} T_{\mu]\nu} = \mathfrak{m}_{\lambda\mu\nu} + \mathfrak{s}_{\lambda\mu\nu}, \end{aligned} \tag{10a}$$

*) For brevity the indices, necessary to distinguish the different components of the field, have been omitted where they are not essential.

where $f_{\lambda\mu\nu}$ according to Dr. Podolanski is defined generally by

$$f_{\lambda\mu\nu} = \tfrac{1}{2}(\mathfrak{f}_{\nu\mu\lambda} + \mathfrak{f}_{\mu\lambda\nu} + \mathfrak{f}_{\nu\lambda\mu}); \qquad (10b)$$

so that

$$f_{\lambda[\mu\nu]} = \mathfrak{f}_{\nu\mu\lambda}. \qquad (11)$$

Then we can express (9), on account of (4), by stating that

$$T_{\mu\nu} = T_{\nu\mu} \qquad (12)$$

is a symmetrical tensor. Since from the definition (10*b*) (where $\mathfrak{f}_{\nu\mu\lambda}$ was antisymmetric in ν and μ) follows

$$f_{\lambda\mu\nu} = - f_{\nu\mu\lambda}, \qquad (13)$$

we find that $\Theta_{\mu\nu}$, defined by (10*a*), satisfies a continuity equation just as $\mathfrak{t}_{\mu\nu}$ did, so that also $T_{\mu\nu}$ satisfies a continuity equation

$$\nabla^{\nu} T_{\mu\nu} = 0. \qquad (14)$$

From (12) and (14) again we conclude that

$$\nabla^{\nu} \mathfrak{j}_{\lambda\mu\nu} = 0. \qquad (15)$$

In view of (9) $\mathfrak{f}_{\lambda\mu\nu}$ might be regarded as the tensor representing the spin density of the field. On the other hand on account of (10*a*) and (15) we may regard $\mathfrak{s}_{\lambda\mu\nu}$ as the spin angular momentum density of the field, $\mathfrak{s}_{\lambda\mu\nu}$ being (unlike $\mathfrak{f}_{\lambda\mu\nu}$) the *moment* of a (spin) *momentum* density $\Theta_{\mu\nu}$. This spin energy-momentum tensor $\Theta_{\mu\nu}$ does not give any contribution to the total energy and momentum, as from (10*a*) and (13) follows

$$\int \Theta_{\mu o} dx_1 dx_2 dx_3 = 0, \quad \text{so} \int T_{\mu o} = \int \mathfrak{t}_{\mu o}. \qquad (16)$$

Thus we have succeeded in defining a spin energy tensor $\Theta_{\mu\nu}$ (to be added to the orbital energy tensor $\mathfrak{t}_{\mu\nu}$) which satisfies a continuity equation, does not give any contribution to the total energy or momentum, but gives a contribution to the angular momentum, ensuring conservation of the total angular momentum density according to (15). The total spin angular momentum is then given by the integral of the corresponding density components over space. Integrating by parts we find on account of (11) and (13)

$$\mathcal{S}_{\mu\nu} \equiv \int (\mathfrak{s}_{\mu\nu o}/c) = \int (\mathfrak{f}_{\mu\nu o}/c) \equiv Re\,\{\int \sum_q p \,.\, S_{[\mu\nu]\,op}\, q\};$$
$$\left(p \equiv \frac{\partial L}{\partial \dot{q}} = - \frac{1}{c} \frac{\partial L}{\partial \nabla^o q}\right). \qquad (17)$$

Thus $\mathfrak{f}_{\mu\nu o}$ as well as $\mathfrak{S}_{\mu\nu o}$ may be used for the calculation of the total spin angular momentum, just as $\mathfrak{t}_{\mu o}$ may be used instead of $T_{\mu o}$ for the calculation of the total energy and momentum (see (16)). In fact the use of $\mathfrak{f}_{\mu\nu o}$ is often easier. It is interesting to remark that x and ∇ do not occur explicitly in the integrand tensor $\mathfrak{f}_{\lambda\mu\nu}$, so that the total spin angular momentum does not depend on the point of reference of the moment. Still $\mathfrak{S}_{\lambda\mu\nu}$ can be regarded as a *regular* angular momentum density (depending on the point of reference), to be added to $\mathfrak{m}_{\lambda\mu\nu}$ in order to find the moment $\mathfrak{j}_{\lambda\mu\nu}$ of the total (= orbital + spin) energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$.

We shall now apply the definitions given by (8) and (10) to all particles and quanta hitherto considered in the literature [5] [6] [7] [8] [9], but for the present we shall suppose that the terms in the Lagrangian function describing the interactions depend on the field components only and not on their gradients.

Expressing the field quantities by means of undors [10] of the first, second, third, ... rank $\Psi_{k_1 k_2 \ldots}$, the terms of the Lagrangian function depending on the derivatives of the field quantities have the form of

$$iK\Psi^{\dagger}B\Gamma^{\lambda}\nabla_{\lambda}\Psi. \tag{18}$$

Here $\Psi^{\dagger}$ is the adjoint ("Hermitean conjugate") of Ψ; B and Γ_λ are given by

$$B = \beta^{(1)}\beta^{(2)}\ldots\ldots \text{ and } \Gamma_\lambda = \varepsilon_1\gamma_\lambda^{(1)} + \varepsilon_2\gamma_\lambda^{(2)} + \ldots.$$

(the $\alpha_\lambda^{(n)}$, $\beta^{(n)}$ and $\gamma_\lambda^{(n)} = \beta^{(n)}\alpha_\lambda^{(n)}$ being Dirac matrices operating on the undor index k_n of Ψ); K is a real normalization factor and $\varepsilon_n = \pm 1$.

Dirac electrons are described by undors of the first rank, Kemmer quanta [6] by undors of the second rank. The quanta of Kemmer's case (*a*) are identical with Pauli-Weisskopf quanta [5]; together with the quanta of Kemmer's case (*c*) they can be described by a Lagrangian function depending on the gradients of the field by (18) with $\varepsilon_1 = -\varepsilon_2$. The quanta of Kemmer's case (*b*) are identical with Proca [7] quanta [2] and with the quanta of "spin = 1" of Dirac [8] and Fierz [9]; together with the quanta of Kemmer's case (*d*) they can be described by taking $\varepsilon_1 = \varepsilon_2$ in (18). The Lagrangian function then has a symmetrical form with respect to both undor indices. The Proca field then is described by the part of Ψ symmetrical in the two indices, and Kemmer's case (*d*) by the antisymmetric part.

The field equations holding for Dirac[8])-Fierz[9]) particles and quanta of "spin > 1" cannot be derived from one single Lagrangian. These fields are described by *symmetrical* undors of a rank $N > 2$. Some of the field equations can be derived from a Lagrangian function depending on the gradients of the field according to (18); the choice of the signs $\varepsilon_n = \pm 1$ is arbitrary except for the condition that $\sum_{n=1}^{N} \varepsilon_n \neq 0$. Further, this Lagrangian should be normalized in a suitable way, so that the right orbital energy density (Hamiltonian function) follows from it. The field equations following from this Lagrangian read

$$\{iNmc/\hbar + \sum_{n=1}^{N} \gamma_\lambda^{(n)} \nabla^\lambda\} \Psi = 0, \text{ (interactions neglected).} \tag{19}$$

In *addition* to (19) other differential equations are assumed by Dirac and Fierz, which, together with (19), can be summarized by writing

$$\{imc/\hbar + \gamma_\lambda^{(n_0)} \nabla^\lambda\} \Psi = 0; \tag{20}$$

n_0 is an arbitrary number $\leq N$. These additional equations cannot be derived from a Lagrangian as the undor Ψ was defined to be symmetrical in its indices. (For the same reason (20) is independent of the choice of n_0). One might say that these equations are introduced in order to make Ψ satisfy a Klein-Gordon equation.

The other possibilities included in (18) have not yet been fully discussed in the literature.

Now, the transformation of the undor Ψ is given by

$$\delta\Psi = \delta\omega^{\mu\nu} S_{\mu\nu op} \Psi, \quad S_{[\mu\nu]op} = \tfrac{1}{4}\Gamma_{[\mu}\Gamma_{\nu]}. \tag{21}$$

Inserting (21) into (8) and (10*b*) we find, on account of

$$B\Gamma_\lambda = \Gamma_\lambda^\dagger B^\dagger, \tag{22}$$

(where $\dagger$ again denotes the adjoint matrix), that

$$f_{\lambda\mu\nu} = Re\,\{\tfrac{1}{4}\, iK \,.\, \Psi^\dagger B\, \Gamma_\lambda \Gamma_\mu \Gamma_\nu \Psi\}. \tag{23}$$

The spin is now given by (17), its density tensor $\mathfrak{s}_{\lambda\mu\nu}$ and the symmetrical energy tensor $T_{\mu\nu}$ by (10*a*). We remark that for undors of the first rank (Dirac wave functions) our method of constructing $T_{\mu\nu}$ is equivalent with the method of symmetrizing the energy tensor for the Dirac electron given by Tetrode[11]) and Pauli[12]).

It should be emphasized that the dependence of the spin of the field on the canonical variables is entirely independent of possible interactions between different kinds of particles and quanta as long as the terms in the Lagrangian function describing these interactions do not contain derivatives of the field components. — On the other hand we made use of the field equations (1) when we were deriving (9) and (12), so that the energy tensor $T_{\mu\nu}$ can only be written explicitly as a symmetrical tensor with the help of these field equations containing all interactions. As one of these equations may be the well known equation of Maxwell's theory

$$\operatorname{div} \vec{\mathfrak{E}} = 4\pi e\rho, \tag{24}$$

which in quantum theory cannot be regarded as a q-number equation but only as a condition imposed on the situation function [13]), in quantum theory the equation $T_{\mu\nu} = T_{\nu\mu}$ may be of the same kind.

§2. We shall now apply the definition (8)—(10) of the spin angular momentum of a field to the case of the meson [3]) [4]) field, that is [14]), an undor field of the second rank $\Psi_{k_1k_2}$ satisfying a wave equation [10]) as given by (19) ($N = 2$), which describes simultaneously (1°) a Proca field $\vec{A}$, V; $\vec{E}$, $\vec{H}$; (the symmetrical part of $\Psi_{k_1k_2}$, case (*b*) of Kemmer [6])); (2°) a field consisting of a pseudo-scalar Y and a pseudo-fourvector $\vec{B}$, W, (case (*d*) of Kemmer), and (3°) a scalar S, which by quantization does not lead to a third type of quanta, as it can be expressed directly in terms of the other canonical variables (in this case these are the variables describing the heavy and the light particles interacting with the meson field). The components Y, $\vec{B}$, W and S together form the antisymmetric part of $\Psi_{k_1k_2}$. The spin angular momentum of this undor field can now be calculated from (17) and (21).

The electric charge density of the mesons is given by [15])

$$e\rho = -\frac{\partial L^{mes}}{\partial \mathfrak{V}} = (e/i\hbar) \cdot \sum_{q\,(mes)} p \cdot q, \tag{25}$$

as the electric scalar potential $\mathfrak{V}$ occurs in L^{mes} only in the combination $\{(\partial/\partial t) - (e/i\hbar)\mathfrak{V}\}$. From (18) we derive [10])

$$e\rho = \Psi^{\dagger} \rho_{op} \cdot e\Psi; \quad \rho_{op} = (K/\hbar c) \cdot B\Gamma^0. \tag{26}$$

Comparing (25) with (26) and with (17), and putting

$$\vec{\mathcal{S}}^{\{N/2\}}_{op} = \tfrac{1}{2}\hbar \sum_{n=1}^{N} \vec{\sigma}^{(n)}_{op}, \quad (\alpha^{(n)}_x \alpha^{(n)}_y = i\sigma^{(n)}_z, \text{ etc.}) \tag{27}$$

we find with the help of (21) and (22)

$$\vec{\mathcal{S}} = \int i\hbar \Psi^{\dagger} \rho_{op} \,.\, \tfrac{1}{4} \Gamma_{[a} \Gamma_{b]} \,.\, \Psi = \int \Psi^{\dagger} \rho_{op} \vec{\mathcal{S}}^{\{1\}}_{op} \Psi. \tag{28}$$

In vector notation, (26) and (28) take the form

$$e\rho = (e/i\hbar) \,.\, \{(\vec{A}^* \,.\, \vec{E}) - (\vec{E}^* \,.\, \vec{A}) + Y^*W - W^*Y\}; \tag{26a}$$

$$\vec{\mathcal{S}} = \int \{[\vec{E}^*, \vec{A}] - [\vec{A}^*, \vec{E}]\}\, dx\, dy\, dz. \tag{28a}$$

From (26*a*) and (25) we recognize that $\vec{A}^*$, $-\vec{E}^*$, Y^* and $-W^*$ are the momenta conjugate to the canonical co-ordinates $\vec{E}$, $\vec{A}$, W and Y respectively. As the latter two do not occur in (28*a*), we conclude that the antisymmetric part of $\Psi_{k_1k_2}$ describes spinless quanta ("singlet states" of the meson field). The Proca field $\vec{A}$, $\vec{E}$, however, represented by the symmetrical part of $\Psi_{k_1k_2}$, possesses a spin angular momentum equal to the expression (28*a*) given already by Proca himself [7]). We may say that Proca quanta represent the "triplet states" of the meson field.

The components $\vec{A}$, $\vec{E}$, W and Y, which are represented by

$$\Psi^I = (\hbar c/2K)^2 \,.\, (\rho_{op})^2\, \Psi, \tag{29}$$

are quantized following the method of Pauli and Weisskopf [5]). According to this method W and Y are expanded in series of scalar waves and $\vec{A}$ and $\vec{E}$ in series of longitudinal and transversal waves. The longitudinal waves are polarized in the direction $\vec{c}^{\,0}_{\vec{p}} = \vec{p}/p$ of the momentum; but the transversal waves we split up into left hand and right hand *circular* polarized waves [16]) characterized by the complex "unity" vectors $\vec{c}^{\,\pm 1}_{\vec{p}}$ defined by

$$\vec{c}^{\,\mu *}_{\vec{p}} = \vec{c}^{\,-\mu}_{\vec{p}}; \quad (\vec{c}^{\,\mu *}_{\vec{p}} \,.\, \vec{c}^{\,\mu'}_{\vec{p}}) = \delta_{\mu\mu'}; \quad [\vec{c}^{\,\varepsilon}_{\vec{p}}, \vec{c}^{\,0}_{\vec{p}}] = i\varepsilon\, \vec{c}^{\,\varepsilon}_{\vec{p}};$$

$$[\vec{c}^{\,-\varepsilon}_{\vec{p}}, \vec{c}^{\,\varepsilon}_{\vec{p}}] = i\varepsilon\, \vec{c}^{\,0}_{\vec{p}}; \quad \vec{c}^{\,-\mu}_{-\vec{p}} = -\vec{c}^{\,\mu}_{\vec{p}} \,.\, (\mu = -1, 0, 1; \; \varepsilon = \pm 1). \tag{30}$$

Further we follow exactly the scheme of Pauli and Weissk o p f [5]) [6]) [15]) [17]). The quantized meson field can then be written in a form analogous to that of the quantized electron field [18]):

$$\Psi^{I}_{k_1k_2} = \sum_{\vec{p}} \{\mathbf{a}_{\vec{p}} X^{(\vec{p})}_{k_1k_2} + \sum_{\mu=-1}^{1} \mathbf{b}_{\vec{p},\mu} \Phi^{(\vec{p},\mu)}_{k_1k_2} + \mathbf{c}^*_{\vec{p}} X^{(\vec{p})£}_{k_1k_2} + \sum_{\mu=-1}^{1} \mathbf{d}^*_{\vec{p},\mu} \Phi^{(\vec{p},\mu)£}_{k_1k_2}\}. \quad (31)$$

Here the $X^{(\vec{p})}_{k_1k_2}$ are antisymmetric undors representing the waves of Y and W; the $\Phi^{(\vec{p},\mu)}_{k_1k_2}$ are symmetrical undors representing the waves of $\vec{A}$ and $\vec{E}$ polarized in the direction of $\vec{c}^{\,\mu}_{\vec{p}}$; the $\Phi^{(\vec{p},\mu)}$ and $X^{(\vec{p})}$ depend on the time and spatial co-ordinates by a factor $\exp\,(i/\hbar)\,(\vec{p}\,.\,\vec{r} - E_p t)$, where $(E_p/c)^2 = (mc)^2 + p^2$. The $\Phi^{(\vec{p},\mu)£}_{k_1k_2}$ and $X^{(\vec{p})£}_{k_1k_2}$ are charge-conjugated [18]) to $\Phi^{(\vec{p},\mu)}_{k_1k_2}$ and $X^{(\vec{p})}_{k_1k_2}$ according to the scheme [10])

$$\Phi^{£}_{k_1k_2} = £^{(1)}£^{(2)}\,\Phi^*_{k_1k_2}; \quad X^{£}_{k_1k_2} = £^{(1)}£^{(2)}\,X^*_{k_1k_2}, \quad (32)$$

where $£^{(1)}$ and $£^{(2)}$ are matrices operating on k_1 and k_2 respectively, satisfying the commutation relations

$$\gamma^{(n)}_{\lambda}£^{(n)} = -\,£^{(n)}\,\gamma^{(n)*}_{\lambda}; \quad £^{(n)}£^{(n)*} = £^{(n)*}\,£^{(n)} = 1. \quad (33)$$

So the $\Phi^{(\vec{p},\mu)£}$ and the $X^{(\vec{p})£}$ depend on the time and spatial co-ordinates by a factor $\exp\,(-\,i/\hbar)\,(\vec{p}\,.\,\vec{r} - E_p t)$.

The functions $X^{(\vec{p})}$, $\Phi^{(\vec{p},\mu)}$, $X^{(\vec{p})£}$ and $\Phi^{(\vec{p},\mu)£}$ are each simultaneously eigenfunctions of the operators $E_{op} = i\hbar\,\partial/\partial t$, $\vec{\mathcal{P}}_{op} = -i\hbar\vec{\nabla}$, e, $e\,.\,(\vec{S}^{\{1\}}_{op})^2$ and $(\vec{S}^{\{1\}}_{op}\,.\,\vec{\mathcal{P}}_{op})\,\vec{\mathcal{P}}_{op}/\mathcal{P}^2_{op}$ belonging to the following eigenvalues:

operator:	eigenfunctions: $X^{(\vec{p})}$	$\Phi^{(\vec{p},\mu)}$	$X^{(\vec{p})£}$	$\Phi^{(\vec{p},\mu)£}$
E_{op}	E_p	E_p	$-E_p$	$-E_p$
$\vec{\mathcal{P}}_{op}$	$\vec{p}$	$\vec{p}$	$-\vec{p}$	$-\vec{p}$
e	e	e	$-(-e)$	$-(-e)$
$e\,S^2_{op}$	0	$e\,.\,2\hbar^2$	0	$-(-e)\,.\,2\hbar^2$
$(\vec{S}.\vec{\mathcal{P}})\vec{\mathcal{P}}/\mathcal{P}^2$	0	$\mu\hbar\,.\,\vec{p}/p$	0	$-\mu\hbar\,.\,\vec{p}/p$

(34)

So these functions form an orthogonal set. Moreover, it turns out that they are quasi-orthogonal in the sense of

$$\int X^{(\vec{p})\dagger}\,\rho_{op}\,X^{(\vec{p}'')} = 0, \qquad (\vec{p} \neq \vec{p}'');$$
$$\int X^{(\vec{p})\dagger}\,\rho_{op}\,\Phi^{(\vec{p}',\mu)} = \int X^{(\vec{p})\dagger}\,\rho_{op}\,X^{(\vec{p}')\pounds} = \int X^{(\vec{p})\dagger}\,\rho_{op}\,\Phi^{(\vec{p}',\mu)\pounds} = 0; \quad (35)$$
$$\int \Phi^{(\vec{p},\mu)\dagger}\,\rho_{op}\,\ldots\ldots = 0; \text{ etc.}$$

If they are quasi-normalized by

$$\int X^{(\vec{p})\dagger}\,\rho_{op}\,X^{(\vec{p}')} = -\int X^{(\vec{p}')\pounds\dagger}\,\rho_{op}\,X^{(\vec{p})\pounds} = \delta(\vec{p}-\vec{p}'),$$
$$\int \Phi^{(\vec{p},\mu)\dagger}\,\rho_{op}\,\Phi^{(\vec{p}',\mu')} = -\int \Phi^{(\vec{p}',\mu')\pounds\dagger}\,\rho_{op}\,\Phi^{(\vec{p},\mu)\pounds} = \delta_{\mu,\mu'}\,\delta(\vec{p}-\vec{p}'), \quad (36)$$

the **a**, **b**, **c*** and **d*** in (31) satisfy the commutation relations[5) 6) 15) 17)]

$$\mathbf{a}_{\vec{p}}\,\mathbf{a}^*_{\vec{p}'} - \mathbf{a}^*_{\vec{p}'}\,\mathbf{a}_{\vec{p}} = \delta(\vec{p}-\vec{p}');$$
$$\mathbf{b}_{\vec{p},\mu}\,\mathbf{b}^*_{\vec{p}',\mu'} - \mathbf{b}^*_{\vec{p}',\mu'}\,\mathbf{b}_{\vec{p},\mu} = \delta_{\mu,\mu'}\,\delta(\vec{p}-\vec{p}');$$
$$\mathbf{c}_{\vec{p}}\,\mathbf{c}^*_{\vec{p}'} - \mathbf{c}^*_{\vec{p}'}\,\mathbf{c}_{\vec{p}} = \delta(\vec{p}-\vec{p}'); \quad (37)$$
$$\mathbf{d}_{\vec{p},\mu}\,\mathbf{d}^*_{\vec{p}',\mu'} - \mathbf{d}^*_{\vec{p}',\mu'}\,\mathbf{d}_{\vec{p},\mu} = \delta_{\mu,\mu'}\,\delta(\vec{p}-\vec{p}'),$$

so that $\mathbf{a}^*_{\vec{p}}\,\mathbf{a}_{\vec{p}}$, $\mathbf{b}^*_{\vec{p},\mu}\,\mathbf{b}_{\vec{p},\mu}$, $\mathbf{c}^*_{\vec{p}}\,\mathbf{c}_{\vec{p}}$ and $\mathbf{d}^*_{\vec{p},\mu}\,\mathbf{d}_{\vec{p},\mu}$ have the eigenvalues 0, 1, 2, 3, ...

From the Lagrangian function

$$L = iK\Psi^\dagger B\,\{2imc/\hbar + \Gamma^\lambda \nabla_\lambda\}\,\Psi + \ldots, \quad (\Gamma_\lambda = \gamma_\lambda^{(1)} + \gamma_\lambda^{(2)}), \quad (38)$$

we derive the energy-momentum density tensor by (2). For the symmetrical tensor representing the current and density of the total (= orbital + spin) energy and momentum we find (neglecting all interactions):

$$T_{\mu\nu} = -(mc/\hbar)\,.\,K\Psi^\dagger\,(\alpha_\mu^{(1)}\alpha_\nu^{(2)} + \alpha_\nu^{(1)}\alpha_\mu^{(2)})\Psi =$$
$$= -(mc/\hbar)\,.\,K\Psi^\dagger B(\gamma_\mu^{(1)}\gamma_\nu^{(2)} + \gamma_\nu^{(1)}\gamma_\mu^{(2)})\Psi, \quad (39)$$

so that the total energy density is given by [19)]

$$T_0{}^0 = 2(mc/\hbar)\,.\,K\Psi^\dagger\Psi. \quad (40)$$

Integrating the (orbital) densities (2) over space we find the total energy and momentum:

$$E = \int \Psi^{\prime\dagger} \rho_{op} E_{op} \Psi', \quad \vec{\mathcal{P}} = \int \Psi^{\prime\dagger} \rho_{op} \vec{\mathcal{P}}_{op} \Psi'. \tag{41}$$

The electric charge is found by integrating (26):

$$e = \int \Psi^{\prime\dagger} \rho_{op}\, e\, \Psi'. \tag{42}$$

In (28), (41) and (42) $\Psi^{\prime\dagger}$ can now be replaced by $\Psi^{I\dagger}$ and Ψ' by Ψ^I, as ρ_{op} is commutative with E_{op}, $\vec{\mathcal{P}}_{op}$, e and $\vec{S}^{\{1\}}_{op}$, while $(\hbar c/2K)^2 \cdot (\rho_{op})^3 = \rho_{op}$. We insert (31) into the expressions for E, $\vec{\mathcal{P}}$, e, and $\vec{S}$, and make use of (34), (35) and (36). Then the minus signs occurring in the latter two colums of (34) are "neutralized" by the minus signs in (36). (In the case of the Dirac electron they are neutralized by the anti-commutativity of the Jordan-Wigner matrices). In this way we find, subtracting infinite zero-point terms:

$$\begin{aligned}
E &= \sum_{\vec{p}} E_p \Big(\mathbf{a}^*_{\vec{p}} \mathbf{a}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{b}^*_{\vec{p},\mu} \mathbf{b}_{\vec{p},\mu} + \mathbf{c}^*_{\vec{p}} \mathbf{c}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{d}^*_{\vec{p},\mu} \mathbf{d}_{\vec{p},\mu}\Big);\\
\vec{\mathcal{P}} &= \sum_{\vec{p}} \vec{p} \Big(\mathbf{a}^*_{\vec{p}} \mathbf{a}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{b}^*_{\vec{p},\mu} \mathbf{b}_{\vec{p},\mu} + \mathbf{c}^*_{\vec{p}} \mathbf{c}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{d}^*_{\vec{p},\mu} \mathbf{d}_{\vec{p},\mu}\Big);\\
e &= \sum_{\vec{p}} \Big\{e\Big(\mathbf{a}^*_{\vec{p}} \mathbf{a}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{b}^*_{\vec{p},\mu} \mathbf{b}_{\vec{p},\mu}\Big) - e\Big(\mathbf{c}^*_{\vec{p}} \mathbf{c}_{\vec{p}} + \sum_{\mu=-1}^{1} \mathbf{d}^*_{\vec{p},\mu} \mathbf{d}_{\vec{p},\mu}\Big)\Big\};\\
\vec{S} &= \sum_{\vec{p}} \sum_{\mu=\pm 1} \mu\hbar \Big\{\vec{c}^{\,o}_{p}\Big(\mathbf{b}^*_{\vec{p},\mu} \mathbf{b}_{\vec{p},\mu} + \mathbf{d}^*_{\vec{p},\mu} \mathbf{d}_{\vec{p},\mu}\Big) - \vec{c}^{\,\mu}_{p}\big(E_p/2mc^2 +\\
&\quad + mc^2/2E_p\big) \cdot \Big(\mathbf{b}^*_{\vec{p},o} \mathbf{b}_{\vec{p},\mu} - \mathbf{b}^*_{\vec{p},-\mu} \mathbf{b}_{\vec{p},o} + \mathbf{d}^*_{\vec{p},o} \mathbf{d}_{\vec{p},\mu} - \mathbf{d}^*_{\vec{p},-\mu} \mathbf{d}_{\vec{p},o}\Big) +\\
&\quad + \vec{c}^{\,\mu}_{p} \big(E_p/2mc^2 - mc^2/2E_p\big) \cdot \Big[\Big(\mathbf{d}_{\vec{p},\mu} \mathbf{b}_{-\vec{p},o} + \mathbf{b}_{\vec{p},\mu} \mathbf{d}_{-\vec{p},o}\Big) \cdot\\
&\quad \cdot \exp(-2iE_pt/\hbar) - \Big(\mathbf{b}^*_{\vec{p},o} \mathbf{d}^*_{-\vec{p},\mu} + \mathbf{d}^*_{\vec{p},o} \mathbf{b}^*_{-\vec{p},\mu}\Big) \cdot \exp(2iE_pt/\hbar)\Big]\Big\}.
\end{aligned} \tag{43}$$

Averaging the expression for $\vec{S}$ with respect to time the terms with **db** $\exp(-2iEt/\hbar)$ and **b*d*** $\exp(2iEt/\hbar)$ vanish indeed; the remaining terms, however, do not depend on the time and do not vanish, contrary to the statement of Durandin and Erschow [2]). From the first term of $\vec{S}$ (43) we conclude that we may regard $\mathbf{b}^*_{\vec{p},\mu} \mathbf{b}_{\vec{p},\mu}$ as the number of positive and $\mathbf{d}^*_{\vec{p},\mu} \mathbf{d}^*_{\vec{p},\mu}$ as the number of negative (triplet) mesons with energy E_p (> 0), momentum $\vec{p}$, and a spin angular momentum component in the direction of this momentum (longitudinal spin component) equal to $\mu\hbar$. The transversal component of the spin angular momentum of the triplet meson is not brought in diagonal form in our representation by longitudinal and circular waves.

In the present paper the superposition of Kemmer's cases (*b*) and (*d*) was considered, as this combination seems to be appropriate for a description of nuclear forces [14].

It is interesting to remark that by an expression like (28*a*) the spin angular momentum $\vec{\mathfrak{S}}_{Maxw}$ of the Maxwellian field is given. Taking as a Lagrangian function for this field

$$4\pi L_{Maxw} = (\mathfrak{H}^2 - \mathfrak{E}^2)/2 - (\vec{\mathfrak{E}} \,.\, \vec{\nabla}\mathfrak{V} + \delta\vec{\mathfrak{A}}/c\delta t) - (\vec{\mathfrak{H}} \,.\, \mathrm{rot}\, \vec{\mathfrak{A}}), \quad (44)$$

we find

$$\vec{\mathfrak{S}}_{Maxw} = (1/4\pi c) \,.\, \int [\mathfrak{E}, \mathfrak{A}]\, dx\, dy\, dz. \quad (45)$$

The common expression for the angular momentum

$$\vec{J}_{Maxw} = (1/4\pi c) \,.\, \int [\vec{r}, [\vec{\mathfrak{E}}, \vec{\mathfrak{H}}]]\, dx\, dy\, dz$$

turns out to be the *total* angular momentum of this field [20]).

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