PHYS-106 (en) Thermodynamics

Week 3 - 05/03/2020

- Probability distribution
  - Discrete distribution
  - Continuous distribution

- Tricks for integral

Discrete distribution

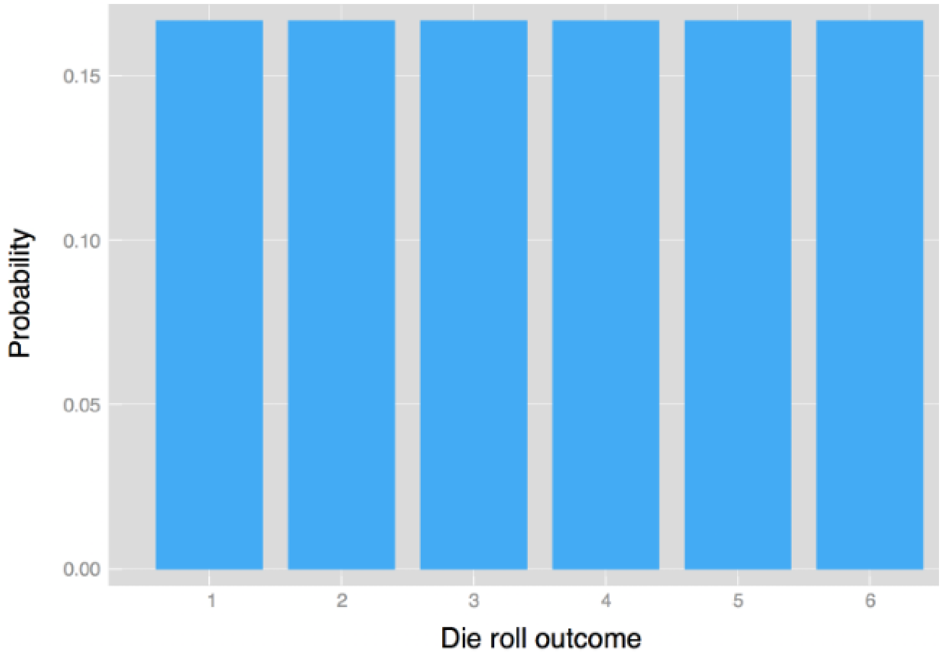
A random variable *X*

- takes **distinct, separate** values
- Example: rolling a die

What number you might get?

$X \sim \{1, 2, 3, 4, 5, 6\}$

- Probability? 1/6
- Distribution



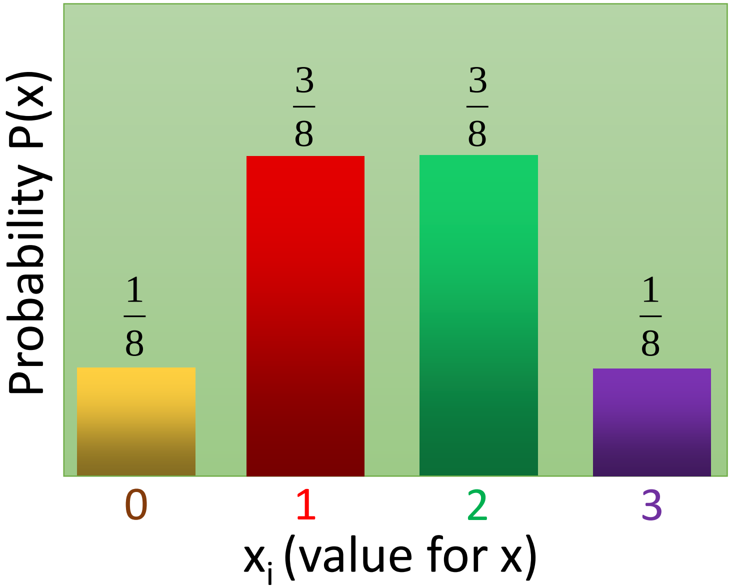
- An experiment of flipping a fair coin

  - X = **#heads** after **3** flips
  - N=8 measurements (y_n): $y_1, y_2, ..., y_8$
  - ➢ K=4 possible **values** (Xi):{0, 1, 2, 3}
  - ➢ Outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

Measurements: y_1=3, y_2=2, y_3=2, y_4=1, y_5=2, y_6=1, y_7=1, y_8=0

- ➢ **Probability**: P(X=0) = 1/8, P(X=1) = 3/8,
  P(X=2) = 3/8, P(X=3) = 1/8

➢Distribution



Probability by counting:

$$P(X_i) = \frac{1}{N}\#(y_n = X_i)$$

The number of measurements where $y_n = X_i$

Mean:

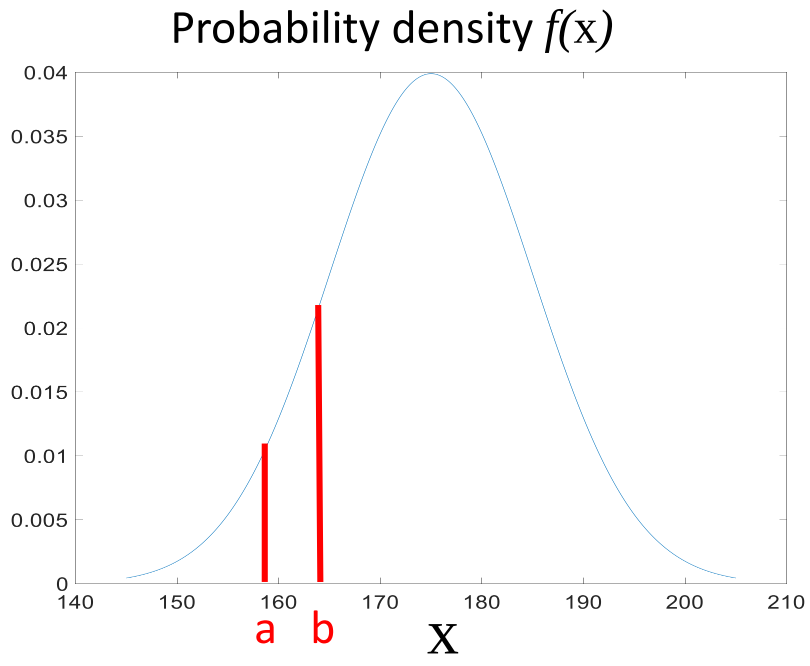
$$\langle X \rangle = \frac{1}{N}\sum_{n=1}^{N} y_n = \sum_{i=1}^{K} X_i\, P(X_i)$$

$$\sum_{i=1}^{K} P(X_i) = 1$$

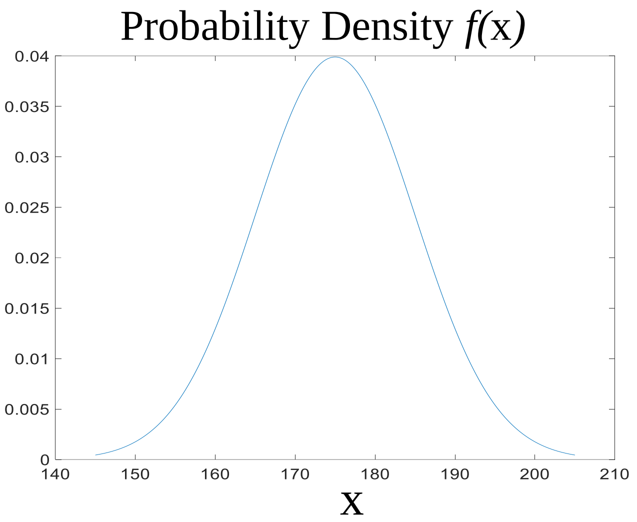
Continuous distribution

- Continuous function
- X takes infinite number of values
- Probability density function (PDF): *f(*x*)*
- P(x) is defined in a interval (a, b):

$$P(\mathrm{x}) = P(a < \mathrm{x} < b) = \int_a^b f(\mathrm{x})dx$$

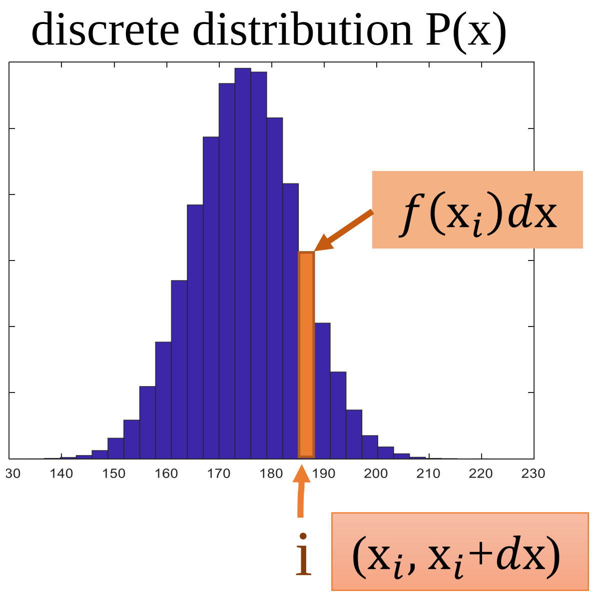


Link to discrete distribution



Discretize x to K bins

K→∞ and dx→0



- Probability for bin i: $P(x_i) = f(x_i)dx$
- Mean:

$$\langle x\rangle = \sum_i^K x_i P(x_i) = \sum_i^K x_i f(x_i)dx$$

- When K→∞ and dx→0, sum is replaced by integral:

$$\langle x\rangle = \int_{-\infty}^{\infty} x f(x)dx$$

- $\int_{-\infty}^{\infty} f(x)dx = 1$

Mean in discrete case:

$$\langle X\rangle = \frac{1}{N}\sum_{n=1}^{N} y_n = \sum_{i=1}^{K} X_i P(X_i)$$

Tricks for integral

- $I_n(a) = \int_0^\infty x^n \exp\left(-ax^2\right)dx$

- n=0, 1, 2, 3, …
- For n = 0, $I_0(a) = \int_0^\infty \exp\left(-ax^2\right)dx$

  Introducing $x' = \sqrt{a}x$

$$I_0(a) = \int_0^\infty \exp\left(-ax^2\right)dx = \frac{1}{\sqrt{a}}\boxed{\int_0^\infty \exp\left(-x'^2\right)dx'}$$

We now consider the following integral:

$$\int_0^\infty \exp\left(-x^2\right)dx \int_0^\infty \exp\left(-y^2\right)dy$$

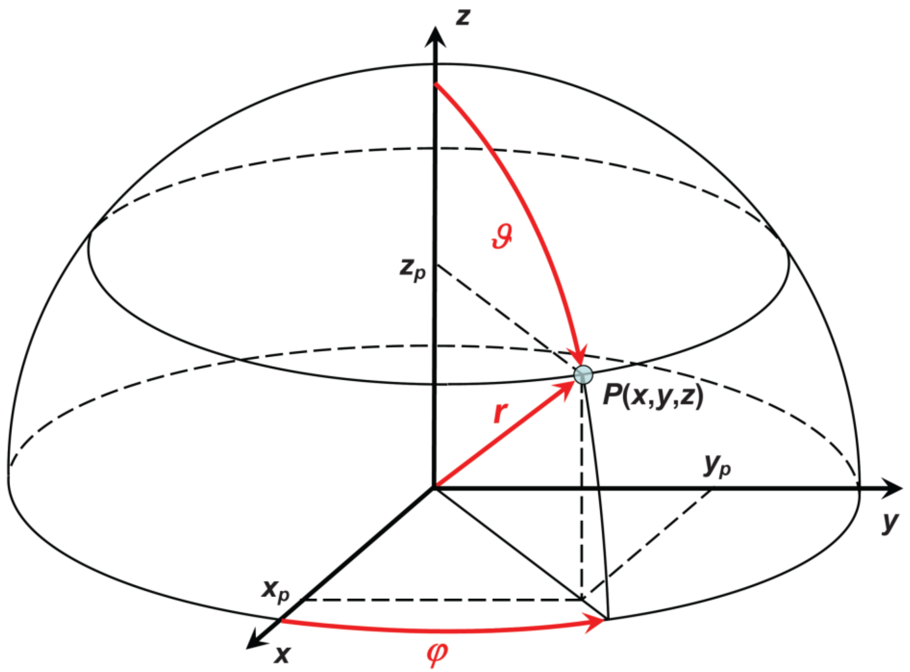
$$= \int_0^\infty \int_0^\infty \exp\left(-x^2\right)\exp\left(-y^2\right)dxdy = \int_0^\infty \int_0^\infty \exp\left[-\left(x^2+y^2\right)\right]dxdy$$

By changing to the polar coordinate system, $x = r\cos\phi$, $y = r\sin\phi$, and $dxdy = rdrd\phi$.
We have:

$$\int_0^\infty \int_0^\infty \exp\left[-\left(x^2+y^2\right)\right]dxdy$$

$$= \frac{1}{4}\int_0^\infty \int_0^{2\pi} r\exp\left(-r^2\right)drd\phi$$

$$= \frac{\pi}{2}\int_0^\infty r\exp\left(-r^2\right)dr = -\frac{\pi}{4}\exp\left(-r^2\right)\Big]_{r=0}^{\infty} = \frac{\pi}{4}$$



Leading to $\int_0^\infty \exp\left(-x^2\right)dx = \frac{\sqrt{\pi}}{2}$

$$I_0(a) = \int_0^\infty \exp\left(-ax^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

- For n = 1,

$$I_1(a) = \int_0^\infty x \exp\left(-ax^2\right) dx$$

$$= -\frac{1}{2a} \exp\left(-ax^2\right)\Big]_0^\infty = \frac{1}{2a}$$

- For n = 2,

$$I_2(a) = \int_0^\infty x^2 \exp\left(-ax^2\right) dx$$

$$= -\frac{d}{da} I_0(a)$$

$$= -\frac{\sqrt{\pi}}{2} \frac{d}{da} a^{-1/2} = \frac{\sqrt{\pi}}{4} a^{-3/2}$$

Exercise: n = 3 and 4

- For n = 3 and 4

$$I_3(a) = \int_0^\infty x^3 \exp\left(-ax^2\right)dx = -\frac{d}{da}I_1 = -\frac{1}{2}\frac{d}{da}a^{-1} = \frac{1}{2}a^{-2}$$

$$I_4(a) = \int_0^\infty x^4 \exp\left(-ax^2\right)dx = -\frac{d}{da}I_2(a) = \frac{\sqrt{\pi}}{4}\frac{d}{da}a^{-3/2} = \frac{3\sqrt{\pi}}{8}a^{-5/2}$$