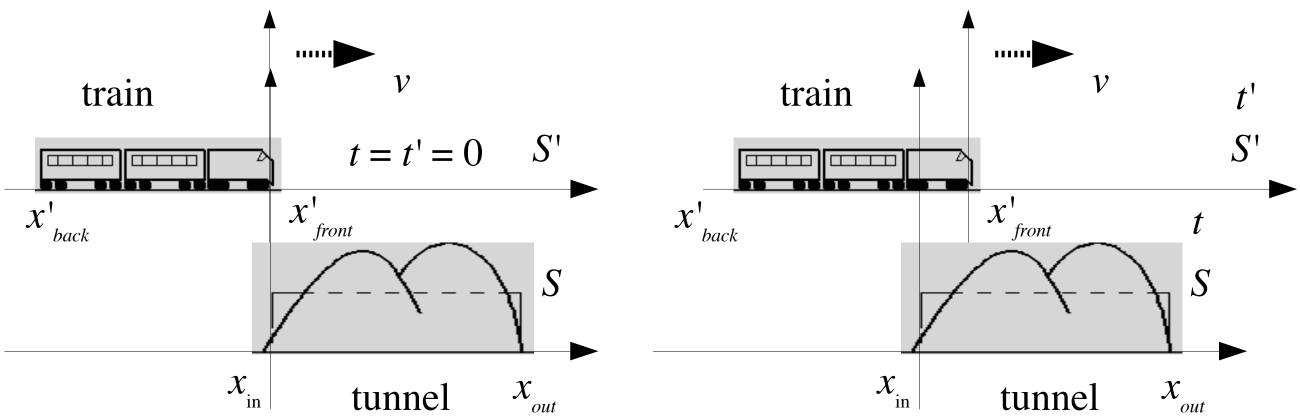
Problem

There is a tunnel with a proper length l_0, and a train with the same proper length l_0. The train is running with a velocity v respect to the tunnel. Consider two frames, S at rest and S' moving in x direction with a velocity v respect to S. Place the tunnel at rest in S frame on the x-axis, where the position of the entrance and exist of the tunnel is given by $x_{in} = 0$ and $x_{out} = x_{in} + l_0 = l_0$, respectively. Similarly the train is placed at rest in S' on the x'-axis and the position of the front and back of the train is given by $x'_{front} = 0$ and $x'_{back} = x'_{front} - l_0 = -l_0$, respectively. At $t = t' = 0$, two fames are overlapping each other. Note that the Lorentz transformation between S and S' is given by

$$\begin{cases} x = \gamma(x' + vt'),\ t = \gamma\left(t' + vx'/c^2\right) \\ x' = \gamma(x - vt),\ t' = \gamma\left(t - vx/c^2\right) \end{cases}.$$



1) Show that at $t_0 = t'_0 = 0$ the train enters in the tunnel in both S and S' frame.
2) Obtain t_1, the time when the train reaches the exit of the tunnel seen in the S frame.
3) What is t'_1, the time for the same event seen in S'? Is the result compatible with time dilatation?
4) Where is the back of the train in S frame at the same time when the front of the train reached the exit of the tunnel seen in the S frame? Is it inside or outside of the tunnel?
5) Is this result compatible with the length contraction?
6) Where is the back of the train seen in S frame at the same time when the front of the train reached the exit of the tunnel seen in the S' frame? Is it inside or outside of the tunnel?
7) Discuss whether the result of 4) and 6) are compatible or not by considering relation between the times of the two incidents in S and S' frames?

Solution

1) The position of the front in S frame at $t_0 = t'_0 = 0$ is given by $x_{front} = \gamma\left(x'_{front} + vt'_0\right) = 0 = x_{in}$, i.e. the position of the train front and that of the tunnel entrance is identical in S frame. Therefore, the train enters in the tunnel at $t_0 = t'_0 = 0$. Similarly, the position of the tunnel entrance seen in S' frame is given by $x'_{in} = \gamma\left(x_{in} - vt_0\right) = 0 = x'_{front}$, i.e. also in S' frame, the position of the train front and that of the tunnel entrance is identical.

2) This is the time needed for the front of the train to move by l_0 in S frame, which is equivalent to the time needed for S' frame to move by l_0 respect to S frame, i.e.

$$t_1 = \frac{l_0}{v}.$$

3) Time t_1 at $x = x_{out}$ in S transform to S' as

$$t_1' = \gamma\left(t_1 - \frac{vx_{out}}{c^2}\right) = \gamma\left(t_1 - \frac{vl_0}{c^2}\right) = \gamma\left(t_1 - \frac{v^2}{c^2}\frac{l_0}{v}\right) = t_1\gamma\left(1 - \frac{v^2}{c^2}\right) = \frac{t_1}{\gamma}.$$

The time dilatation gives a relation between the time interval measured by a clock at rest, Δt_0, i.e. proper time, and that measured by a moving clock, Δt, as $\Delta t = \gamma \Delta t_0$. In our case, t_1' is measured by the clock in the train where the observer is in the train, i.e. the clock is at rest and $t_1' - t_0' = t_1'$ is proper time corresponding to Δt_0. On the other hand, $t_1 - t_0 = t_1$ is measured by the clock moving from x_{in} to x_{out} with a velocity v while the observer is at rest. Therefore, the relation between t_1 and t_1' is in agreement with time dilatation.

4) The position of the back of the train at t_1 in S is given by $x_{back} = \gamma\left(x'_{back} + v\tilde{t}_1'\right)$, where $\tilde{t}_1'$ is t_1 at x_{back} seen in S', i.e. $\tilde{t}_1' = \gamma\left(t_1 - vx_{back}/c^2\right)$. It follows that

$$\begin{aligned} x_{back} &= \gamma\left[x'_{back} + v\gamma\left(t_1 - \frac{vx_{back}}{c^2}\right)\right] \\ &= \gamma\left(-l_0 + v\gamma\frac{l_0}{v} - v\gamma\frac{vx_{back}}{c^2}\right) \\ &= -\gamma l_0 + \gamma^2 l_0 - \gamma^2\frac{v^2}{c^2}x_{back}. \end{aligned}$$

Using

$$x_{back} + \gamma^2\frac{v^2}{c^2}x_{back} = \left(1 + \frac{1}{1 - v^2/c^2}\frac{v^2}{c^2}\right)x_{back} = \gamma^2 x_{back}$$

x_{back} can be obtained as

$$x_{back} = l_0\left(1 - \frac{1}{\gamma}\right).$$

Since $\gamma > 1$, $x_{back} > 0$ and the end of the train is already inside of the tunnel.

5) The length of train seen in the S frame is given by $x_{front} - x_{back} = l_0 - l_0(1 - 1/\gamma) = l_0/\gamma$. The length contraction is given by the relation between the length of an object at rest, Δd_0, i.e. proper length and the length of the same object moving with a velocity v, $\Delta d = \Delta d_0/\gamma$. In our case, l_0 is measured at rest, i.e. proper length and $x_{front} - x_{back}$ is the length of the train moving with a velocity v measured at the same time. Therefore, the result is compatible with the length contraction.

6) The position of the back of the train seen in S when the front of the train reached at the exit of the tunnel in S' is given by $x_{back} = \gamma\left(x'_{back} + vt_1'\right)$, since the train reached at the exit of the tunnel at t_1' in S'. It follows that

$$\begin{aligned} x_{back} &= \gamma\left(x'_{back} + vt_1'\right) \\ &= \gamma\left(-l_0 + v\frac{t_1}{\gamma}\right) \\ &= l_0(1 - \gamma) \end{aligned}$$

Since $\gamma > 1$, $x_{back} < 0$ and the end of the train is still outside of the tunnel.

7) The two events are not the same, i.e. x_{back} calculated for 4) is associated with time t_1 while x_{back} given in 6) is associated with time

$$\hat{t}_1 = \gamma\left(t_1' + vx_{back}'/c^2\right) = \gamma\left(\frac{t_1}{\gamma} - \frac{vl_0}{c^2}\right) = t_1 - \gamma\frac{l_0}{v}\frac{v^2}{c^2} = t_1\left(1 - \gamma\frac{v^2}{c^2}\right) < t_1 .$$

Since they occur not at the same time in *S*, the results of 4) and 6) do not need to agree. Since $\hat{t}_1 < t_1$, seen in the S frame, the front of the train reaches the exit of the tunnel in the S' frame first then the head of the train reaches the exit of the tunnel in S frame. Then, the end of the train is still outside at $\hat{t}_1$ then later at t_1 it is inside in the S frame, which is not in contradiction.