General Physics II: Tutorial Material 4

1) In an inertial frame *S*, two objects, *a* with its rest mass m_a and *b* with its rest mass m_b, are moving with velocity vectors $\vec{v}_a$ and $\vec{v}_b$, respectively, with θ being the angle between the two vectors. After colliding each other, *a* and *b* got stack each other and became one object *x* with its rest mass M and velocity $\vec{V}$.

   a) Write down the relativistic energy-momentum relations among *a*, *b* and *x*.

Energy and momentum relations among the three objects are given by

$$E_x = E_a + E_b \text{ and } \vec{p}_x = \vec{p}_a + \vec{p}_b$$

where relativistic energies and momenta are

$$\begin{aligned} E_a &= \sqrt{m_a^2c^4 + p_a^2c^2} \\ E_b &= \sqrt{m_b^2c^4 + p_b^2c^2} \\ E_x &= \sqrt{M^2c^4 + p_x^2c^2} \end{aligned} \quad \text{and} \quad \begin{aligned} \vec{p}_a &= \frac{m_a\vec{v}_a}{\sqrt{1-\beta_a^2}} \\ \vec{p}_b &= \frac{m_b\vec{v}_b}{\sqrt{1-\beta_b^2}} \\ \vec{p}_x &= \frac{M\vec{V}}{\sqrt{1-\beta_V^2}} \end{aligned}$$

Note that the following notations are used: $\vec{\beta} = \vec{v}/c$, $\beta = (|\vec{v}|/c)$ and $p = |\vec{p}|$.

   b) Calculate *M* as a function of m_a, m_b, $|\vec{v}_a|$, $|\vec{v}_b|$ and θ.

From the energy momentum relations

$$E_x^2 = E_a^2 + E_b^2 + 2E_aE_b \text{ and } p_x^2 = p_a^2 + p_b^2 + 2p_ap_b\cos\theta$$

and relativistic energy

$$E_x^2 = M^2c^4 + p_x^2c^4$$

it follows that

$$\begin{aligned} M^2 &= \frac{1}{c^4}\left(E_x^2 - p_x^2c^2\right) \\ &= \frac{1}{c^4}\left(E_a^2 + E_b^2 + 2E_aE_b\right) - \frac{1}{c^2}\left(p_a^2 + p_b^2 + 2p_ap_b\cos\theta\right) \end{aligned}$$

where θ is the angle between $\vec{v}_a$ and $\vec{v}_b$ as defined previously. It follows that

$$\begin{aligned} M^2 &= m_a^2 + m_b^2 + 2m_am_b\sqrt{1+\frac{p_a^2}{c^2m_a^2}}\sqrt{1+\frac{p_b^2}{c^2m_b^2}} - \frac{2p_ap_b}{c^2}\cos\theta \\ &= m_a^2 + m_b^2 + \frac{2m_am_b}{\sqrt{1-\beta_a^2}\sqrt{1-\beta_b^2}} - \frac{2m_am_b\beta_a\beta_b}{\sqrt{1-\beta_a^2}\sqrt{1-\beta_b^2}}\cos\theta \\ &= m_a^2 + m_b^2 + \frac{2m_am_b}{\sqrt{1-\beta_a^2}\sqrt{1-\beta_b^2}}\left(1-\beta_a\beta_b\cos\theta\right) \end{aligned}$$

c) Show that when $\vec{v}_a = \vec{v}_b$, we have $M = m_a + m_b$.

If $\vec{v}_a = \vec{v}_b \equiv \vec{v}$ and $\theta = 0$, which leads to

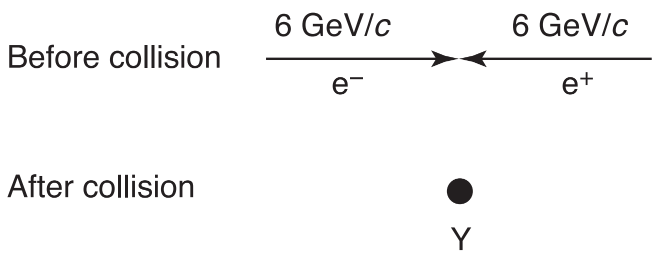
$$M^2 = m_a{}^2 + m_b{}^2 + \frac{2m_a m_b}{1-\beta^2}\left(1-\beta^2\right)$$
$$= \left(m_a + m_b\right)^2$$

where $\beta = |\vec{v}|/c$, thus, $M = m_a + m_b$.

d) Explain the result obtained for 3), from the principle of relativity.

If we move to a new inertial frame, *S*', which is moving with a velocity $\vec{v}$ respect to *S*, in *S*' two objects, *a* and *b*, are both at rest, thus also *c*. Energies of the three bodies in the S' frame are given by their rest mass, which leads to $M = m_a + m_b$.

2) An electron, e^-, and its anti-particle, a positron, e^+, collide head-on with an equal momentum $p_0 = 6\ \text{GeV}/c$. The rest masses of the electron and positron are equal and about 0.0005 GeV/c^2. Therefore, they are totally negligible in the energy $E = \sqrt{m_0^2 c^4 + p^2 c^2}$, leading to $E^- = p_0 c = 6\ \text{GeV}$ and $E^+ = p_0 c = 6\ \text{GeV}$ for e^- and e^+ energies, respectively.

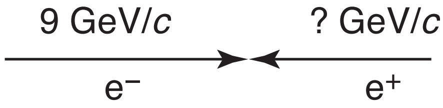


a) When they collide, the electron and positron annihilate and one new particle, Y, is produced. Using the energy-momentum conservation law, calculate the momentum and rest mass of Y in the GeV/c and GeV/c^2 units, respectively.

By choosing the direction of the *x*-axis to be aligned with the electron direction, momentum vectors for the initial electron and positron are given by $(p_0, 0, 0)$ and $(-p_0, 0, 0)$. Therefore, the total momentum of the initial state is 0 and the total energy $2p_0 c$. From the momentum conservation, Y is produced at rest, thus its energy is given by $E^Y = m_0^Y c^2$ where m_0^Y is the rest mass of Y. From the energy conservation, it follows that $m_0^Y c^2 = 2p_0 c$. Therefore, the rest mass of Y is given by

$$m_0^Y = \frac{2p_0}{c} = \frac{12\ \text{GeV}/c}{c} = \frac{12\ \text{GeV}}{c^2}$$

b) We collide an e^- with a momentum of 9 GeV/c head-on with an e^+. What is the momentum of e^+ in order to produce after the collision only one Y particle, identical to that in a)? In which direction does the particle Y move and how large is $\beta_u = u/c$, where u is the velocity of Y? The rest mass of e^- and e^+ can be neglected in the calculations.



By denoting the momentum of the positron to be p'^+, the momentum vector of the e^-e^+ initial state is given by $\left(9\ \mathrm{GeV}/c - p'^+,\ 0,\ 0\right)$ and the energy $9\ \mathrm{GeV} + p'^+c$. Due to the momentum conservation, the y- and z-components of the Y momentum vector are 0 and for x-component, we obtain $9\ \mathrm{GeV}/c - p'^+ = p'^{\mathrm{Y}}$, where p'^{Y} is the x-component of the Y momentum vector. From the energy conservation, we obtain

$$9\ \mathrm{GeV} + p'^+c = E'^{\mathrm{Y}} = \sqrt{m_0^{\mathrm{Y}2}c^4 + p'^{\mathrm{Y}2}c^2},$$

where E'^{Y} is the energy of Y. Using the momentum conservation, we obtain

$$\sqrt{m_0^{\mathrm{Y}2}c^4 + p'^{\mathrm{Y}2}c^2} = \sqrt{144\ \mathrm{GeV}^2 + (9\ \mathrm{GeV} - p'^+c)^2}$$

It follows that

$$\left(9\ \mathrm{GeV} + p'^+c\right)^2 = 144\ \mathrm{GeV}^2 + \left(9\ \mathrm{GeV} - p'^+c\right)^2$$

thus

$$36\ \mathrm{GeV}\, p'^+c = 144\ \mathrm{GeV}^2$$

leading to the positron momentum to be

$$p'^+ = 4\frac{\mathrm{GeV}}{c}$$

The momentum and energy of Y is

$$p'^{\mathrm{Y}} = 9\ \mathrm{GeV}/c - p'^+ = 5\ \mathrm{GeV}/c$$

and

$$E'^{\mathrm{Y}} = 9\ \mathrm{GeV} + p'^+c = 13\ \mathrm{GeV},$$

respectively. Therefore, Y moves the direction of the electron.

By recalling the energy and momentum of a particle moving with a velocity of u and with a rest mass of m_0

$$E = \frac{m_0c^2}{\sqrt{1-(u/c)^2}},\ p = \frac{m_0u}{\sqrt{1-(u/c)^2}}$$

it follows that

$$\frac{cp}{E} = \frac{m_0uc}{\sqrt{1-(u/c)^2}} \Bigg/ \frac{m_0c^2}{\sqrt{1-(u/c)^2}} = \frac{u}{c}$$

For Y, we then obtain

$$\beta_u = \frac{u}{c} = \frac{cp'^{\mathrm{Y}}}{E'^{\mathrm{Y}}} = \frac{5}{13} = 0.38$$

3) A certain galaxy has a Doppler shift given by $f_0 - f = 0.0987\, f_0$. Estimate how fast it is moving away from us.

$$f = 0.9013f_0 = \sqrt{\frac{c-u}{c+u}}f_0,\ \ u = 0.1035c$$

4) Show that when $u \ll c$, the Doppler shift in wavelength is

$$\frac{\Delta\lambda}{\lambda_0} = \frac{u}{c}$$

For $u \ll c$

$$\lambda = \lambda_0\sqrt{\frac{c+u}{c-u}} = \lambda_0\sqrt{\frac{1+u/c}{1-u/c}} \approx \lambda_0\sqrt{1+2u/c} \approx \lambda_0(1+u/c)$$

thus

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda-\lambda_0}{\lambda_0} \approx \frac{\lambda_0(1+u/c)-\lambda_0}{\lambda_0} = \frac{u}{c}$$