General Physics II: Tutorial Material 3

1) An unstable particle is at rest and suddenly decays into two fragments. No external forces act on the particle or its fragments. One of the fragments has a velocity of $0.60c$ and a mass of $4\ \mathrm{GeV}/c^2$, while the other has a mass of $1\ \mathrm{GeV}/c^2$. What is the speed of the less massive fragment?

The momenta of the lighter and heavier particles in the rest frame of the initial particle are denoted by p_1 and p_2, respectively, and $p_1 = p_2$, i.e.

$$\frac{m_1 u_1}{\sqrt{1-(u_1/c)^2}} = \frac{m_2 u_2}{\sqrt{1-(u_2/c)^2}} \quad \to \quad u_1^2 = \frac{F}{m_1^2 + F/c^2}, \quad F = \frac{m_2^2 u_2^2}{1-(u_2/c)^2}$$

Thus.

$u_1^2 = 0.9\ c^2$, i.e. $u_1 \approx 0.95\ c$.

2) The rest mass of an electron is $m_e = 500\ \mathrm{keV}/c^2$. What is the velocity of an electron, u_e (in the unit of the speed of light), whose momentum is $p_e = 375\ \mathrm{keV}/c$? What is the kinetic energy, K_e, of this electron?

The momentum of the electron is given by

$$p = \frac{m_0 u}{\sqrt{1-(u/c)^2}}, \text{ thus } u^2 = \frac{p^2 c^2}{p^2 + m_0^2 c^2}, \text{ giving, } u_e^2 = 0.36\ c^2, \text{ i.e. } u_e \approx 0.6\ c.$$

The total energy is given by

$$E = \frac{m_0 c^2}{\sqrt{1-(u/c)^2}} = 625\ \mathrm{KeV}, \text{ and kinetic energy } K_e = E - mc^2 = 125\ \mathrm{KeV}$$

3) A particle was created in a laboratory.

i) It took time, t, to move a distance, L, which is usually called flight length, and then decayed. How long did the particle live in the frame where the particle was at rest?

The time particle lived in the frame, t_0, where the particle is at rest is clearly the proper time interval. Therefore, it is related to that measured in the laboratory frame, t, as

$$t = \frac{t_0}{\sqrt{1-(u/c)^2}}$$

where u is the velocity of the particle in the laboratory frame. At the same time, the velocity of the particle in the laboratory frame is given by $u = L/t$. It follows that

$$t_0 = t\sqrt{1-\left(\frac{L}{ct}\right)^2}$$

ii) In a real experiment, we measure the energy, E, the momentum, p, and the flight length, L, in the laboratory. Calculate how long the particle lived in the frame where the particle was at rest.

Let us denote $\beta_u = u/c$, where u is the velocity of the particle in the laboratory frame and m_0 to be the rest mass. From the relativistic definition of energy and momentum

$$E = \frac{m_0 c^2}{\sqrt{1-\beta_u^2}}, p = \frac{m_0 u}{\sqrt{1-\beta_u^2}}$$

we obtain

$$\beta_u = \frac{cp}{E}$$

and the time the particle lived in the laboratory frame is given by $t = L/(c\beta_u)$. It follows that

$$t_0 = t\sqrt{1-\beta_u^2} = \frac{LE}{c^2 p}\sqrt{1-\left(\frac{cp}{E}\right)^2} = \frac{L}{c^2 p}\sqrt{E^2 - c^2 p^2}$$

where t_0 is the time the particle at rest lived.

4) A beam of protons is injected to the Large Hadron Collider at CERN with a kinetic energy of 450 GeV and accelerated to 4.5 TeV, i.e. the kinetic energy is increased by a factor of 10. What is the increase of the velocity? Is it as much as the kinetic energy? If not, why?

From

$$T = \frac{m_0 c^2}{\sqrt{1-(u/c)^2}} - m_0 c^2$$

we can derive

$$\left(\frac{u}{c}\right)^2 = \frac{T^2 + 2Tm_0c^2}{T^2 + 2Tm_0c^2 + m_0^2c^4}$$

Since the kinetic energy is much higher than the rest energy,

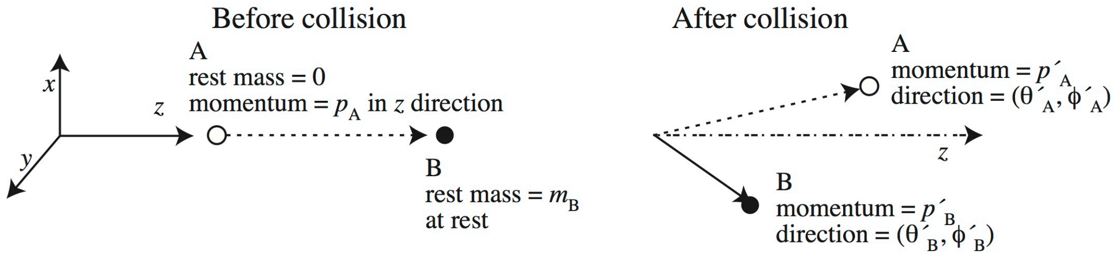
$$\left(\frac{u}{c}\right)^2 = \frac{T^2 + 2Tm_0c^2}{T^2 + 2Tm_0c^2 + m_0^2c^4} \approx 1 - \frac{m_0^2c^4}{T^2 + 2Tm_0c^2} \approx 1 - \frac{m_0^2c^4}{T^2}$$

thus

$$\left(\frac{u}{c}\right) \approx 1 - \frac{1}{2}\frac{m_0^2c^4}{T^2}$$

For the kinetic energy of 450 GeV, we have $u/c = 0.9999975$, and for 4.5 TeV $u/c = 0.999999975$, i.e. the velocity is increased by only 1.0000025. The kinetic energy is used to increase the mass, not the velocity.

5) Consider a case where a particle A, which is mass less, moves along the z-axis with a momentum p_A collides with a particle B at rest, whose rest mass is m_B. After the collision, Particle A moves with a momentum p'_A into a direction (θ'_A, ϕ'_A), and B with p'_B and (θ'_B, ϕ'_B), where θ and ϕ are the polar and azimuth angles, respectively.



1) From the momentum conservation in the x- and y- components between before and after the collision, show that $\phi'_A - \phi'_B = \pi$, i.e. the momentum vectors of the two particles are in the same plane, and that it leads to $p'_A \sin\theta'_A - p'_B \sin\theta'_B = 0$.

The x-, y-, and z-components of the momentum and energy for A and B before are given by

$$p_x^A = 0,\ p_y^A = 0,\ p_z^A = p_A,\ E^A = p_A c$$
$$p_x^B = 0,\ p_y^B = 0,\ p_z^B = 0,\ E^B = m_B c^2$$

thus the components of the total momentum and energy before the collision are

$$p_x^{\text{Before}} = 0,\ p_y^{\text{Before}} = 0,\ p_z^{\text{Before}} = p_A,\ E^{\text{Before}} = m_B c^2 + p_A c\,.$$

Similar for the after the collision, for A and B,

$$p_x'^A = p'_A \sin\theta'_A \cos\phi'_A,\ p_y'^A = p'_A \sin\theta'_A \sin\phi'_A,\ p_z'^A = p'_A \cos\theta'_A,\ E'_A = p'_A c$$
$$p_x'^B = p'_B \sin\theta'_B \cos\phi'_B,\ p_y'^B = p'_B \sin\theta'_B \sin\phi'_B,\ p_z'^B = p'_B \cos\theta'_B,\ E'_B = \sqrt{m_B^2 c^4 + p_B'^2 c^2}$$

and for the total

$$p_x^{\text{After}} = p'_A \sin\theta'_A \cos\phi'_A + p'_B \sin\theta'_B \cos\phi'_B,$$
$$p_y^{\text{After}} = p'_A \sin\theta'_A \sin\phi'_A + p'_B \sin\theta'_B \sin\phi'_B,$$
$$p_z^{\text{After}} = p'_A \cos\theta'_A + p'_B \cos\theta'_B$$
$$E^{\text{After}} = p'_A c + \sqrt{m_B^2 c^4 + p_B'^2 c^2}$$

The energy momentum conservation between the before and after the collision leads to

$$p'_A \sin\theta'_A \cos\phi'_A + p'_B \sin\theta'_B \cos\phi'_B = 0$$
$$p'_A \sin\theta'_A \sin\phi'_A + p'_B \sin\theta'_B \sin\phi'_B = 0$$
$$p'_A \cos\theta'_A + p'_B \cos\theta'_B = p_A$$
$$p'_A c + \sqrt{m_B^2 c^4 + p_B'^2 c^2} = p_A c + m_B c^2$$

The first two equations correspond to the conservation of x and y momentum components. By multiplying the first equation by $\sin\phi'_B$ and second by $\cos\phi'_B$, they become

$$p'_A \sin\theta'_A \cos\phi'_A \sin\phi'_B + p'_B \sin\theta'_B \cos\phi'_B \sin\phi'_B = 0$$
$$p'_A \sin\theta'_A \sin\phi'_A \cos\phi'_B + p'_B \sin\theta'_B \sin\phi'_B \cos\phi'_B = 0\,.$$

By subtracting one from the other, it follows that

$$p'_A \sin\theta'_A (\cos\phi'_A \sin\phi'_B - \sin\phi'_A \cos\phi'_B) = -p'_A \sin\theta'_A \sin(\phi'_A - \phi'_B) = 0$$

thus $\sin(\phi'_A - \phi'_B) = 0$, i.e. $\phi'_A - \phi'_B = 0$ or $= \pi$. From the first two equations, we also get

$$\begin{aligned}
&(p'_A \sin\theta'_A \cos\phi'_A + p'_B \sin\theta'_B \cos\phi'_B)^2 + (p'_A \sin\theta'_A \sin\phi'_A + p'_B \sin\theta'_B \sin\phi'_B)^2 \\
&= (p'_A \sin\theta'_A)^2 + (p'_B \sin\theta'_B)^2 + 2p'_A p'_B \sin\theta'_A \sin\theta'_B (\cos\phi'_A \cos\phi'_B + \sin\phi'_A \sin\phi'_B) \\
&= (p'_A \sin\theta'_A)^2 + (p'_B \sin\theta'_B)^2 + 2p'_A p'_B \sin\theta'_A \sin\theta'_B \cos(\phi'_A - \phi'_B) = 0
\end{aligned}.$$

Since $(p'_A \sin\theta'_A)^2 > 0$, $(p'_B \sin\theta'_B)^2 > 0$, and azimuth angles are 0 to π, thus $p'_A p'_B \sin\theta'_A \sin\theta'_B > 0$, it must be $\cos(\phi'_A - \phi'_B) < 0$. By combining the two conditions, we obtain $\phi'_A - \phi'_B = \pi$, leading to

$$\begin{aligned}
&(p'_A \sin\theta'_A)^2 + (p'_B \sin\theta'_B)^2 + 2(p'_A p'_B \sin\theta'_A \sin\theta'_B)\cos(\phi'_A - \phi'_B) \\
&= (p'_A \sin\theta'_A)^2 + (p'_B \sin\theta'_B)^2 - 2p'_A p'_B \sin\theta'_A \sin\theta'_B \\
&= (p'_A \sin\theta'_A - p'_B \sin\theta'_B)^2 \\
&= 0
\end{aligned}$$

thus $p'_A \sin\theta'_A - p'_B \sin\theta'_B = 0$.

2) By combining the energy conservation and momentum conservation, show that

$$\frac{1}{p'_A} - \frac{1}{p_A} = \frac{2}{m_B c}\sin^2\left(\frac{\theta'_A}{2}\right).$$

With $\phi'_A - \phi'_B = \pi$, the energy momentum conservation is now given by

$$\left.\begin{aligned}
p'_A \sin\theta'_A - p'_B \sin\theta'_B &= 0 \\
p'_A \cos\theta'_A + p'_B \cos\theta'_B &= p_A
\end{aligned}\right\} \quad \text{momentum conservation}$$

$$p'_A c + \sqrt{m_B^2 c^4 + p'^2_B c^2} = p_A c + m_B c^2 \quad \text{energy conservation}.$$

By rewriting the energy conservation as

$\sqrt{m_B^2 c^4 + p'^2_B c^2} = (p_A c - p'_A c) + m_B c^2$, we can remove the √ operation as

$$m_B^2 c^4 + p'^2_B c^2 = (p_A c - p'_A c)^2 + m_B^2 c^4 + 2(p_A c - p'_A c) m_B c^2$$

leading to

$$-2(p_A c - p'_A c) m_B c^2 = (p_A c - p'_A c)^2 - p'^2_B c^2.$$

Using the two equations for momentum conservation, p'_B can be expressed by p_A, p'_A, and θ'_A, as

$$\begin{aligned}
p'^2_B &= (p'_A \sin\theta'_A)^2 + (p_A - p'_A \cos\theta'_A)^2 \\
&= p_A^2 + p'^2_A - 2p_A p'_A \cos\theta'_A
\end{aligned}.$$

By combining the two results,

$$-2(p_A c - p'_A c) m_B c^2 = (p_A c - p'_A c)^2 - p_A^2 c^2 - p'^2_A c^2 + 2 p_A p'_A c^2 \cos\theta'_A$$
$$= -2 p_A p'_A c^2 (1 - \cos\theta'_A).$$

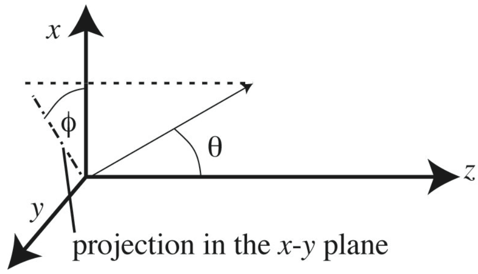
Using $1 - \cos\theta'_A = 1 - \cos^2(\theta'_A/2) + \sin^2(\theta'_A/2) = 2\sin^2(\theta'_A/2)$, we obtain

$$p_A - p'_A = \frac{p_A p'_A}{m_B c} 2\sin^2\left(\frac{\theta'_A}{2}\right)$$

and by dividing both side by $p_A p'_A$,

$$\frac{1}{p'_A} - \frac{1}{p_A} = \frac{2}{m_B c}\sin^2\left(\frac{\theta'_A}{2}\right).$$

NB: Polar and azimuth angles are defined as



You may recall $\sin(\phi \pm \theta) = \sin\phi\cos\theta \pm \cos\phi\sin\theta$ and $\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$.