# General Physics II: Tutorial Material 13

1) Heat pump is used to warm up a room at temperature $T_1$ by transferring thermal energy from outside at temperature $T_2$, where $T_1 > T_2$ i.e. the outside is colder than the room, using work done to the heat pump. Show that a heat pump is more economical than heating the room directly with the work by computing the efficiency of the heat pump using the Carnot cycle.

If we operate the Carnot cycle in the reversed order, A→D→C→B→A, heats for D→C and B→A, $Q_{\mathrm{dc}}$ and $Q_{\mathrm{ba}}$, respectively, are given by

$$Q_{\mathrm{dc}} = nRT_2 \ln\frac{V_{\mathrm{c}}}{V_{\mathrm{d}}} > 0 \text{ and } Q_{\mathrm{ba}} = nRT_1 \ln\frac{V_{\mathrm{a}}}{V_{\mathrm{b}}} < 0$$

and similarly for the work, A→D, D→C, C→B, and B→A

$$W_{\mathrm{ad}} = \frac{P_{\mathrm{b}}V_{\mathrm{b}}}{1-\gamma}\left[\left(\frac{V_{\mathrm{c}}}{V_{\mathrm{b}}}\right)^{1-\gamma} - 1\right],\ W_{\mathrm{dc}} = nRT_2 \ln\frac{V_{\mathrm{c}}}{V_{\mathrm{d}}},\ W_{\mathrm{cb}} = \frac{P_{\mathrm{b}}V_{\mathrm{b}}}{1-\gamma}\left[1 - \left(\frac{V_{\mathrm{c}}}{V_{\mathrm{b}}}\right)^{1-\gamma}\right],\ W_{\mathrm{ba}} = nRT_1 \ln\frac{V_{\mathrm{d}}}{V_{\mathrm{c}}}.$$

The total work is then given by

$$W_{\mathrm{total}} = W_{\mathrm{ad}} + W_{\mathrm{dc}} + W_{\mathrm{cb}} + W_{\mathrm{ba}} = nR(T_2 - T_1)\ln\frac{V_{\mathrm{b}}}{V_{\mathrm{a}}}$$

The efficiency of a heat pump given as

$$\varepsilon_{\text{heat pump}} = \frac{\text{thermal energy given to the heat reservoir with } T = T_1}{\text{total work given to the heat pump}}$$

leads to

$$\varepsilon_{\text{heat pump}} = \frac{-Q_{\mathrm{ba}}}{-W_{\mathrm{total}}} = \frac{-nRT_1 \ln\frac{V_{\mathrm{a}}}{V_{\mathrm{b}}}}{-nR(T_2 - T_1)\ln\frac{V_{\mathrm{b}}}{V_{\mathrm{a}}}} = \frac{T_1}{T_1 - T_2} > 1.$$

Since $\varepsilon_{\text{heat pump}} > 1$, heat pump works more efficient than converting directly the work given to the heat pump, $W_{\mathrm{total}}$, directly to the thermal energy to heat the room.

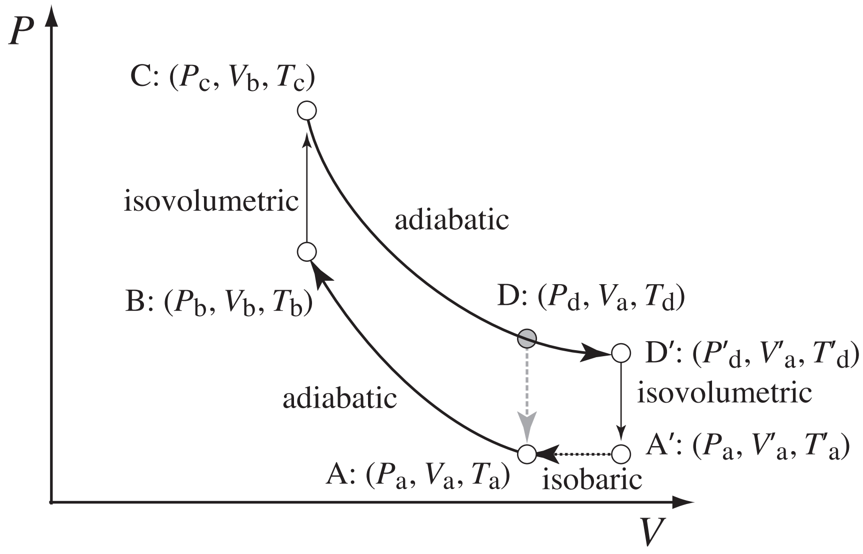
2) In the Otto cycle, the volume ratio in the expansion, C→D, is identical to that for the compression, A→B, and is given by $V_a/V_b$. Some hybrid cars use Atkinson cycle where the volume ratios are different. This is realised by changing the timing of exhaust or/and intake and also called Miller cycle. In order to compare its performance with the Otto cycle, we consider the Miller cycle to use the same volume of air-gasoline mixture gas, $V_a$, for the adiabatic compression and the condition for ignition, i.e. the adiabatic compression of the air-gasoline mixture starts at $A(P_{\mathrm{a}}, V_{\mathrm{a}}, T_{\mathrm{a}})$.
The Miller cycle shown in the *P*-*V* plot below is the following:
i) At A, the piston is somewhere in the middle of the cylinder. The volume, $V_a$, is filled with the air-gasoline mixture and all the valves are closed.
ii) The piston moves up to the top (B) and the gas is ignited and explodes (B→C).
iii) The piston is pushed down (C→D') and reaches the lowest position of the cylinder (D').

iv) The piston goes up to exhaust the burnt gas (exhaust valve open) and goes down to take in the air-gasoline mixture (intake valve open), which corresponds to the isovolumetric reduction of the pressure, D'→A', where at A' the cylinder is back at the lowest position.

v) When the piston starts to move up, the intake valve is still open, thus isobaric compression starts till arriving at A where the in-take valve closes.

vi) Back to the original state and ready for the next cycle.

Figure below is the *P*-*V* plot for a Miller cycle, together with an equivalent Otto cycle.




Show that an engine with Miller cycle (A→B→C→D'→A'→A) is more efficient than that with Otto cycle (A→B→C→D→A).

The figure above shows the *P*-*V* plots for the Miller cycle (A→B→C→D'→A'→A) and Otto cycle (A→B→C→D→A) to be compared. Since the initial volume and pressure of the air-gasoline mixtures are same, we assume that the two cycles star with the same amount of gas molecule from A.

As discussed previously, the area surrounded by a cycle on the *P*-*V* plane gives the total work. Therefore, the total work for the Otto cycle, $W_{\text{total}}^{\text{Otto}}$, is less than that of the Miller cycle, $W_{\text{total}}^{\text{Miller}}$, i.e. $W_{\text{total}}^{\text{Otto}} < W_{\text{total}}^{\text{Miller}}$.

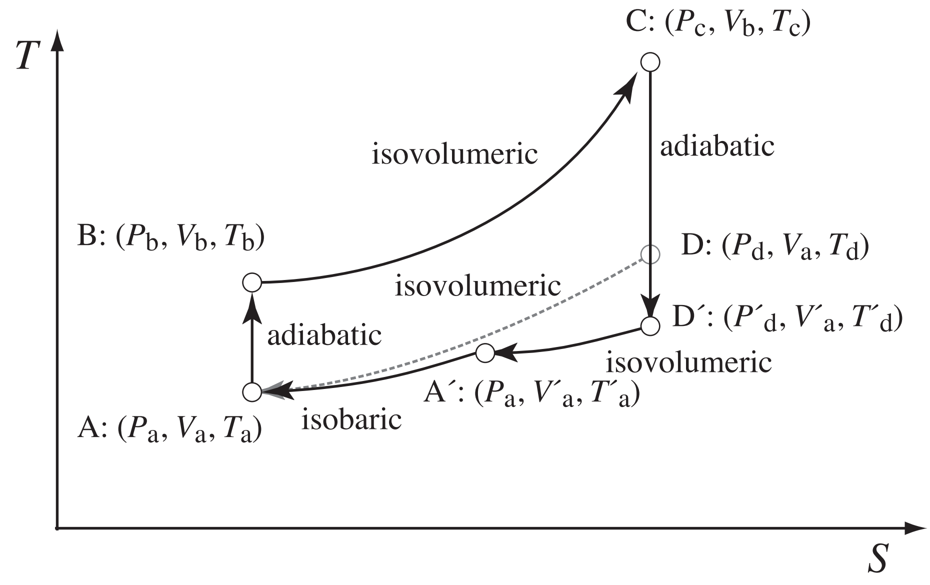
Now we consider the entropy. Entropy changes for A→B, C→D and C→D' are zero, for B→C, D→A and D'→A',

$$\Delta S_{\text{bc}} = nC_{\text{V}} \ln \frac{T_{\text{c}}}{T_{\text{b}}},\ \Delta S_{\text{da}} = nC_{\text{V}} \ln \frac{T_{\text{a}}}{T_{\text{d}}},\ \Delta S'_{\text{da}} = nC_{\text{V}} \ln \frac{T'_{\text{a}}}{T'_{\text{d}}}$$

and for A'→A,

$$\Delta S'_{\text{aa}} = nC_{\text{P}} \ln \frac{T_{\text{a}}}{T'_{\text{a}}}$$

Since $P_{\text{b}} < P_{\text{c}}$, and from $PV = nRT$ we have $T_{\text{b}} < T_{\text{c}}$, thus $\Delta S_{\text{bc}} > 0$. Similarly, from $P_{\text{a}} < P_{\text{d}}$, $P_{\text{a}} < P'_{\text{d}}$ and $V_{\text{a}} < V'_{\text{a}}$ we obtain $T_{\text{a}} < T_{\text{d}}$, $T'_{\text{a}} < T'_{\text{d}}$ and $T_{\text{a}} < T'_{\text{a}}$, thus $\Delta S_{\text{da}} < 0$, $\Delta S'_{\text{da}} < 0$ and $\Delta S'_{\text{aa}} < 0$. From $PV^{\gamma}$ = constant in the adiabatic process and $PV = nRT$ lead to $T_{\text{a}} < T_{\text{b}}$ and $T_{\text{d}} < T_{\text{c}}$. It follows that the *S*-*T* plots for the Otto cycle (A→B→C→D→A) and Miller cycle (A→B→C→D'→A'→A) can be drawn as




The thermal energy flow into the engine, $Q_{\text{in}}$, is the positive heat in the system given by the area below the B→C line for both Otto and Miller cycles, thus $Q_{\text{in}}^{\text{Otto}} = Q_{\text{in}}^{\text{Miller}}$.
The efficiency of an engine is given by

$$\varepsilon = \frac{W_{\text{total}}}{Q_{\text{in}}}$$

From $W_{\text{total}}^{\text{Otto}} < W_{\text{total}}^{\text{Miller}}$ and $Q_{\text{in}}^{\text{Otto}} = Q_{\text{in}}^{\text{Miller}}$, the Miller cycle is more efficient.

3) Two questions related to thermal conductivity and Fourier's law for the heat flow rate

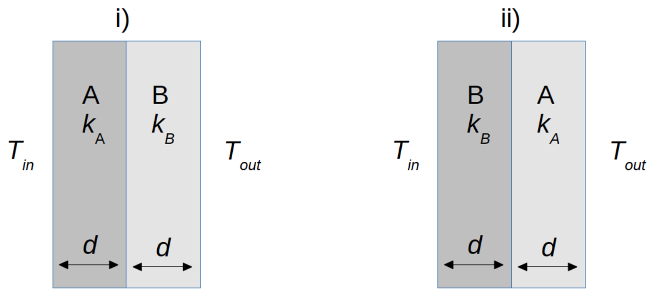
$$\dot{Q} \equiv \frac{dQ}{dt} = -kA\frac{dT}{dx}$$

I) Let us consider a wall consisting of two plates, A and B: both plates have a thickness $d$ with thermal conductivities, $k_A$ and $k_B$, respectively, and $k_A > k_B$. We use this wall for a house and can make i) surface A facing inside of the house and ii) surface B facing inside of the house. In winter when the outside temperature, $T_{out}$, is lower than the room temperature $T_{in}$, i.e. $T_{in} > T_{out}$.

   a. Calculate the heat flow rate from the room to outside through wall with configuration i).

   b. Calculate the heat flow rate from the room to outside through wall with configuration ii).

   c. Are the temperature profiles through the wall from the inside to the outside surface for the two configurations same or different? Which configuration loses more thermal energy to outside?




Fourier's law for a surface area $A$, can be written as

$$\dot{Q}dx = -kAdT \ .$$

Place the inner surface of the wall at $x = 0$ and denote the temperature at the boarder of the two layers to be $T_m$ .

***a)*** For Configuration i), integrations of the above equation over $x$ for the left side and $T$ for the right side give, for the first layer and second layer, to be

$$\dot{Q}\int_0^d dx = -k_a A\int_{T_{in}}^{T_m} dT \text{ and } \dot{Q}\int_d^{2d} dx = -k_b A\int_{T_m}^{T_{out}} dT$$

and leads to

$$\dot{Q} = \frac{-k_a A\left(T_m - T_{in}\right)}{d} \text{ and } \dot{Q} = \frac{-k_b A\left(T_{out} - T_m\right)}{d}$$

Since the two heat rates must be identical, $T_m$ can be obtained as

$$k_a\left(T_m - T_{in}\right) = k_b\left(T_{out} - T_m\right)$$
$$T_m\left(k_a + k_b\right) = k_a T_{in} + k_b T_{out}$$
$$T_m = \frac{k_a T_{in} + k_b T_{out}}{k_a + k_b}$$

Inserting this to the expression of Q, we obtain

$$\dot{Q} = -\frac{k_a k_b}{k_a + k_b}\frac{A}{d}\left(T_{out} - T_{in}\right)$$

Note that for $k_a = k_b \equiv k$ , it gives

$$\dot{Q} = -\frac{kA\left(T_{out} - T_{in}\right)}{2d}$$

i.e. identical to a $2d$ thick wall with a thermal conductivity $k$.

***b)*** In a similar manner, we obtain for the Configuration ii)

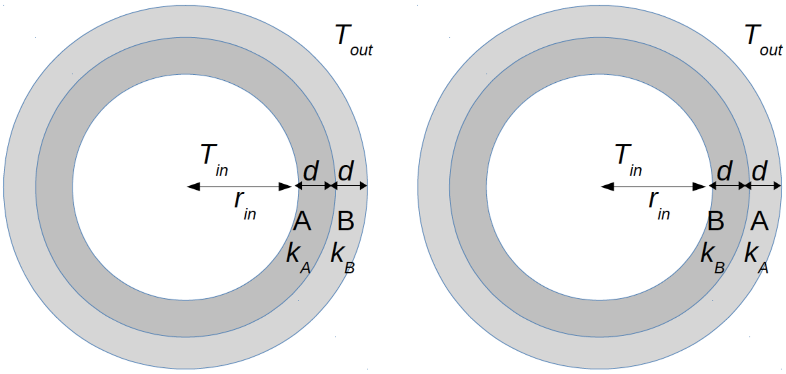
$$T_m = \frac{k_b T_{in} + k_a T_{out}}{k_a + k_b}$$

and

$$\dot{Q} = -\frac{k_a k_b}{k_a + k_b}\frac{A}{d}\left(T_{out} - T_{in}\right)$$

***c)*** The temperature at the border of the two layers, $T_m$, is different for the two configurations, thus the temperature profiles in the wall are different. Since $k_a > k_b$ and $T_{in} > T_{out}$, the $T_m$ is higher for the Configuration i), i.e. the temperature drops slower in the first layer in Configuration i) than in ii). However, the heat rates are identical for the two configurations. Therefore, the loss of thermal energy in the two configurations are identical.

II) A pipe consists of the two layers of material with a same thickness $d$. The inner radius of the pipe is $r_{in}$. Two material A and B with thermal conductivities, $k_A$ and $k_B$, respectively, are available for the layers where $k_A > k_B$. This pipe is used to transport hot water with a temperature $T_{in}$ through cold outside with a temperature of $T_{out}$, where $T_{in} > T_{out}$. Figures below show the cross-sections of the pipes.




a. Calculate the heat rate from the water to outside for a pipe where the inner layer with material A.

b. Calculate the heat rate from the water to outside for a pipe where the inner layer with material B.

c. Are the radial temperature profiles different between the two configurations? Which configuration loses more thermal energy to outside?

For a pipe, i.e. a cylindrical geometry, Fourier's law becomes

$$\dot{Q}\frac{dr}{r} = -2\pi k l dT$$

where $l$ is the length of the pipe.

***a)*** For configuration i), the heat rate for the first layer is given by the integration of the above formula as

$$\dot{Q}\int_{r_{in}}^{r_{in}+d}\frac{dr}{r} = -2\pi kl\int_{T_{in}}^{T_m} dT$$

$$\dot{Q} = \frac{-2\pi k_a l}{\ln\left[\left(r_{in}+d\right)/r_{in}\right]}\left(T_m - T_{in}\right)$$

and for the second layer

$$\dot{Q}\int_{r_{in}}^{r_{in}+d}\frac{dr}{r} = -2\pi kl\int_{T_{in}}^{T_m} dT$$

$$\dot{Q} = \frac{-2\pi k_b l}{\ln\left[\left(r_{in}+2d\right)/\left(r_{in}+d\right)\right]}\left(T_{out} - T_m\right)$$

where $T_m$ is the temperature at the border of the two layers. Since the two heat rates must be identical

$$\frac{k_a}{\ln\left[\left(r_{in}+d\right)/r_{in}\right]}\left(T_m - T_{in}\right) = \frac{k_b}{\ln\left[\left(r_{in}+2d\right)/\left(r_{in}+d\right)\right]}\left(T_{out} - T_m\right)$$

and $T_m$ can be obtained as

$$T_m = \frac{\dfrac{k_a T_{in}}{\ln\left(1+\delta\right)} + \dfrac{k_b T_{out}}{\ln\left[\left(1+2\delta\right)/\left(1+\delta\right)\right]}}{\dfrac{k_a}{\ln\left(1+\delta\right)} + \dfrac{k_b}{\ln\left[\left(1+2\delta\right)/\left(1+\delta\right)\right]}}$$

$$= \frac{k_a T_{in}\ln\left(1+2\delta\right) + \left(k_b T_{out} - k_a T_{in}\right)\ln\left(1+\delta\right)}{k_a\ln\left(1+2\delta\right) + \left(k_b - k_a\right)\ln\left(1+\delta\right)}$$

where $\delta = d/r_{in}$. By inserting $T_m$ to the expression of $\dot{Q}$, we obtain

$$\dot{Q} = -\frac{2\pi l k_a k_b}{k_a\ln\left(1+2\delta\right) + \left(k_b - k_a\right)\ln\left(1+\delta\right)}\left(T_{out} - T_{in}\right)$$

***b)*** Similarly for the Configuration ii), we obtain

$$T_m = \frac{k_b T_{in}\ln\left(1+2\delta\right) + \left(k_a T_{out} - k_b T_{in}\right)\ln\left(1+\delta\right)}{k_b\ln\left(1+2\delta\right) + \left(k_a - k_b\right)\ln\left(1+\delta\right)}$$

$$\dot{Q} = -\frac{2\pi l k_a k_b}{k_b\ln\left(1+2\delta\right) + \left(k_a - k_b\right)\ln\left(1+\delta\right)}\left(T_{out} - T_{in}\right)$$

***c)*** Two configurations have different $T_m$. Therefore the temperature profiles in the radial direction are different for the two different configurations of pipes. The difference in the thermal energy lost to outside is given by

$$
\begin{aligned}
\left|\dot{Q}_1\right| - \left|\dot{Q}_2\right| &= \frac{2\pi l k_a k_b}{k_a \ln(1+2\delta) + (k_b - k_a)\ln(1+\delta)}(T_{in} - T_{out}) - \frac{2\pi l k_a k_b}{k_b \ln(1+2\delta) + (k_a - k_b)\ln(1+\delta)}(T_{in} - T_{out}) \\
&= 2\pi l k_a k_b (T_{in} - T_{out}) \frac{k_b \ln(1+2\delta) + (k_a - k_b)\ln(1+\delta) - k_a \ln(1+2\delta) + (k_b - k_a)\ln(1+\delta)}{\left[k_a \ln(1+2\delta) + (k_b - k_a)\ln(1+\delta)\right] k_b \ln(1+2\delta) + (k_a - k_b)\ln(1+\delta)} \\
&= \frac{2\pi l k_a k_b (T_{in} - T_{out})(k_a - k_b)\left[2\ln(1+\delta) - \ln(1+2\delta)\right]}{\left[k_a \ln(1+2\delta) + (k_b - k_a)\ln(1+\delta)\right]\left[k_b \ln(1+2\delta) + (k_a - k_b)\ln(1+\delta)\right]}
\end{aligned}
$$

Let us consider a function $f(\delta) = 2\ln(1+\delta) - \ln(1+2\delta)$. At $\delta = 0$ we have $f(0) = 0$. The first derivative of $f(\delta)$ is given by

$$
\begin{aligned}
\frac{df(\delta)}{d\delta} &= \frac{2}{1+\delta} - \frac{2}{1+2\delta} \\
&= \frac{2\delta}{(1+\delta)(1+2\delta)}
\end{aligned}
$$

which is >0 for $\delta > 0$, thus $f(\delta)$ is a monotonically increasing function for $\delta > 0$ and we have $f(\delta) > 0$ for $\delta > 0$. In conclusion, $\left|\dot{Q}_1\right| - \left|\dot{Q}_2\right| > 0$, i.e. more thermal energy is lost to outside for the configuration 1.