General Physics II: Tutorial Material 12

1) The temperature of n-mol ideal gas has changed from T_1 to T_2 degrees. Determine the entropy change for 1) under constant pressure and 2) under constant volume.

Under constant pressure, heat is given by $Q = nC_P \Delta T$ or $\tilde{d}Q = nC_P dT$. Entropy change is then given by

$$\Delta S = \int_{T_1}^{T_2} \frac{\tilde{d}Q}{T} = nC_P \int_{T_1}^{T_2} \frac{dT}{T} = nC_P \ln\frac{T_2}{T_1}$$

Similarly for the constant volume, we obtain

$$\Delta S = \int_{T_1}^{T_2} \frac{\tilde{d}Q}{T} = nC_V \int_{T_1}^{T_2} \frac{dT}{T} = nC_V \ln\frac{T_2}{T_1}.$$

Those relations can also be obtained from the entropy change for the general case

$$\Delta S = nC_V \ln\frac{T_2}{T_1} + nR\ln\frac{V_2}{V_1}$$

shown before. If the volume is constant, $V_1 = V_2$ and

$$\Delta S = nC_V \ln\frac{T_2}{T_1}$$

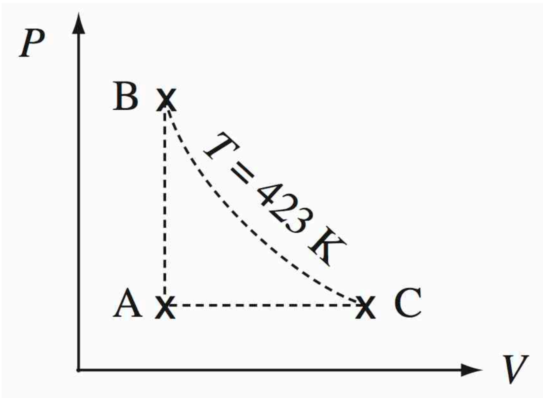
If the pressure is constant, from $PV = nRT$, it follows that

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = nRT$$

and

$$\Delta S = nC_V \ln\frac{T_2}{T_1} + nR\ln\frac{V_2}{V_1} = n\left(C_V + R\right)\ln\frac{T_2}{T_1} = nC_P \ln\frac{T_2}{T_1}$$

2) Figure below is the V-P diagram of a heat engine with 1 mol of a diatomic molecule ideal gas. At point A, it is at STP (273 K and 1 atm). Points B and C are on the isothermal line at $T = 423$ K. The process A-B is with a constant volume and A-C with a constant pressure.



a) Obtain the volume, pressure and temperature for the state B and C.

The initial state A is given by

$$\mathrm{A}\left(T = 273\ \mathrm{K},\ P = 1.013\times10^5\ \mathrm{N/m^2},\ V = \frac{nRT}{P} = 2.24\times10^{-2}\ \mathrm{m^3}\right)$$

The temperature of the state B is given as T=423 K, and it is on the isovolumetric line with A, we have $V_{\mathrm{B}} = V_{\mathrm{A}}$ and

$$\mathrm{B}\left(T = 423\ \mathrm{K},\ P = \frac{nRT}{V} = 1.57 \times 10^{5}\ \mathrm{N/m^2},\ V = 2.24 \times 10^{-2}\ \mathrm{m^3}\right)$$

The state C is on the isobaric line with A, i.e. $P_{\mathrm{C}} = P_{\mathrm{A}}$, and on the isothermal line with B, i.e. $T_{\mathrm{C}} = T_{\mathrm{B}}$, thus we have

$$\mathrm{C}\left(T = 423\ \mathrm{K},\ P = 1.013 \times 10^{5}\ \mathrm{N/m^2},\ V = \frac{nRT}{P} = 3.47 \times 10^{-2}\ \mathrm{m^3}\right)$$

b) Which is the path to generate the work, A-B-C or A-C-B, and why?

By noting that no work is done in A→B or B→A, since they are isovolumetric processes, the work done for the cycle A→B→C→A is given by the area below B-C minus the area below A-C on the V-P diagram, which is positive. The work done for the cycle A→C→B→A is given by the area below A-C minus the area below B-C, which is negative. Therefore A→B→C→A is the cycle generates the work.

c) What is the efficiency, ε, of the engine where ε = W / Q (positive)?

The work done in the isothermal path, B→C, is given by

$$W_{\mathrm{B}\to\mathrm{C}} = nRT\ln\frac{V_{\mathrm{C}}}{V_{\mathrm{B}}} = 1\ \mathrm{mol} \times 8.314\ \mathrm{J/mol\cdot K} \times 423\ \mathrm{K} \times \ln\frac{3.47 \times 10^{-2}}{2.24 \times 10^{-2}} = 1539\ \mathrm{J}$$

and C→A

$$W_{\mathrm{C}\to\mathrm{A}} = P_{\mathrm{A}}\left(V_{\mathrm{A}} - V_{\mathrm{C}}\right) = 1.013 \times 10^{5}\ \mathrm{N/m^2} \times (2.24 - 3.47) \times 10^{-2}\ \mathrm{m^3} = -1246\ \mathrm{J}$$

Therefore the total work, i.e. net-work done, is

$$W = W_{\mathrm{B}\to\mathrm{C}} + W_{\mathrm{C}\to\mathrm{A}} = 1539\ \mathrm{J} - 1246\ \mathrm{J} = 293\ \mathrm{J}$$

For the heat, processes with positive heat are A→B and B→C. Using the molar specific heat under constant volume is given by $C_{\mathrm{V}} = n_{\mathrm{f}}R/2$, where the number of degrees of freedom, $n_{\mathrm{f}} = 5$ for a diatomic molecule, the heat for A→B is given as

$$Q_{\mathrm{A}\to\mathrm{B}} = nC_{\mathrm{V}}\Delta T = 1\ \mathrm{mol} \times \frac{5}{2} \times 8.314\ \mathrm{J/mol\cdot K} \times (423 - 273)\ \mathrm{K} = 3118\ \mathrm{J}.$$

No change in the internal energy is generated in B→C, thus

$$Q_{\mathrm{B}\to\mathrm{C}} = W_{\mathrm{B}\to\mathrm{C}} = 1539\ \mathrm{J}$$

making to total positive heat into the system to be

$$Q_{\mathrm{in}} = Q_{\mathrm{A}\to\mathrm{B}} + Q_{\mathrm{B}\to\mathrm{C}} = 4657\ \mathrm{J}$$

Thus the efficiency of the engine is given by

$$\varepsilon = \frac{W}{Q_{\mathrm{in}}} = 0.063$$

i.e. about 6%.

d) Show that total heat minus total work is zero.

Using the molar specific heat under constant pressure, $C_\mathrm{P} = C_\mathrm{V} + R$, the heat for C→A is given by

$$Q_{\mathrm{A}\to\mathrm{B}} = nC_\mathrm{P}\Delta T = 1\ \mathrm{mol} \times \frac{7}{2} \times 8.314\ \mathrm{J/mol\cdot K} \times (273 - 423)\ \mathrm{K} = -4365\ \mathrm{J}$$

Therefore, the sum of the total heat and work is given by

$$Q_{\mathrm{A}\to\mathrm{B}} + Q_{\mathrm{B}\to\mathrm{C}} + Q_{\mathrm{C}\to\mathrm{A}} - W_{\mathrm{B}\to\mathrm{C}} - W_{\mathrm{C}\to\mathrm{A}} = 0\ \mathrm{J}\ (\text{within the rounding error})$$

3) We consider now a similar heat engine starting from A as defined above, but the B-C path is done adiabatically. The temperature of B is kept at $T = 423$ K and on the isovolumetric line with A. The state C remains on the isobaric line with A.

a) Obtain the volume, pressure and temperature of C.

The states A and B are given as before

$$\mathrm{A}\left(T = 273\ \mathrm{K},\ P = 1.013 \times 10^5\ \mathrm{N/m^2},\ V = 2.24 \times 10^{-2}\ \mathrm{m^3}\right)$$

and

$$\mathrm{B}\left(T = 423\ \mathrm{K},\ P = 1.57 \times 10^5\ \mathrm{N/m^2},\ V = 2.24 \times 10^{-2}\ \mathrm{m^3}\right)$$

The state C is still on the isobaric line with A, i.e. $P_\mathrm{C} = P_\mathrm{A}$, but now on the adiabatic line with B, i.e. $V_\mathrm{C}{}^\gamma P_\mathrm{C} = V_\mathrm{B}{}^\gamma P_\mathrm{B}$. By recalling that $V_\mathrm{A} = V_\mathrm{B}$, we obtain

$$V_\mathrm{C} = V_\mathrm{B}\left(\frac{P_\mathrm{B}}{P_\mathrm{C}}\right)^{1/\gamma} = V_\mathrm{A}\left(\frac{P_\mathrm{B}}{P_\mathrm{A}}\right)^{1/\gamma}$$

and

$$T_\mathrm{C} = \frac{P_\mathrm{C}V_\mathrm{C}}{nR} = \frac{P_\mathrm{A}V_\mathrm{A}}{nR}\left(\frac{P_\mathrm{B}}{P_\mathrm{A}}\right)^{1/\gamma}$$

For the diatomic ideal gas, we have $C_\mathrm{V} = 5R/2$ and $C_\mathrm{P} = C_\mathrm{V} + R = 7R/2$, leading to

$$\frac{1}{\gamma} = \frac{C_\mathrm{V}}{C_\mathrm{P}} = 0.714$$

i.e.

$$\mathrm{C}\left(T = 373.2^\circ\ \mathrm{K},\ P = 1.013 \times 10^5\ \mathrm{N/m^2},\ V = 3.063 \times 10^{-2}\ \mathrm{m^3}\right)$$

b) Calculate the efficiency.

As for the previous engine, nor work for A→B and for C→A

$$W_{\mathrm{C}\to\mathrm{A}} = P_\mathrm{A}(V_\mathrm{A} - V_\mathrm{C}) = 1.013 \times 10^5\ \mathrm{N/m^2} \times (2.24 - 3.06) \times 10^{-2}\ \mathrm{m^3} = -830\ \mathrm{J}$$

For the work in B→C, using the adiabatic relation, $V^\gamma P = \text{constant} = V_\mathrm{B}{}^\gamma P_\mathrm{B}$, it follows that

$$W_{\mathrm{C}\to\mathrm{A}} = \int_B^C P\mathrm{d}V = V_\mathrm{B}^\gamma P_\mathrm{B}\int_B^C V^{-\gamma}\mathrm{d}V = \frac{V_\mathrm{B}^\gamma P_\mathrm{B}}{1-\gamma}V^{1-\gamma}\Big]_{V_\mathrm{B}}^{V_\mathrm{C}} = \frac{V_\mathrm{B}^\gamma P_\mathrm{B}}{1-\gamma}\left(V_\mathrm{C}^{1-\gamma} - V_\mathrm{B}^{1-\gamma}\right)$$

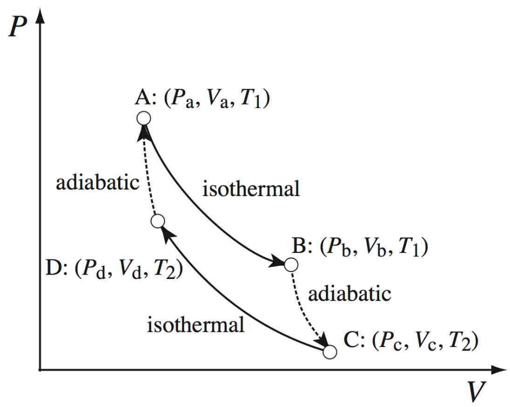
and $1 - \gamma = -0.4$ giving $W_{\mathrm{B}\to\mathrm{C}} = 1034$ J. The total work is then

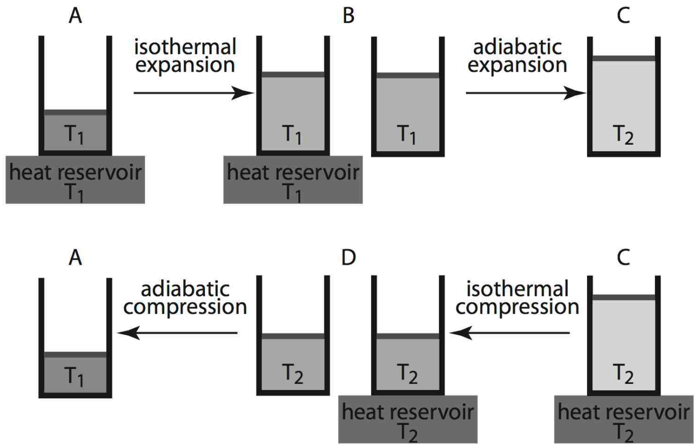
$$W = W_{\mathrm{B}\to\mathrm{C}} + W_{\mathrm{C}\to\mathrm{A}} = 204\ \mathrm{J}$$

For the heat, $Q_{A\to B} = 3118$ J is unchanged from the previous case, but no heat in B→C since it is an adiabatic process. Therefore, total positive heat is $Q_{\mathrm{in}} = Q_{A\to B} = 3118$ J and efficiency

$$\varepsilon = \frac{W}{Q_{\mathrm{in}}} = 0.065$$

4) Calculate the change of the total entropy of the Carnot cycle after one cycle, i.e. that of the Carnot engine plus the two heat reservoirs.





As described in the lecture note, thermal energy flows into the engine from the two heat reservoirs are given by

$$Q_1 = W_{\mathrm{ab}} = nRT_1 \ln\frac{V_{\mathrm{b}}}{V_{\mathrm{a}}} > 0 \text{ and } Q_2 = W_{\mathrm{cd}} = nRT_2 \ln\frac{V_{\mathrm{d}}}{V_{\mathrm{c}}} < 0$$

in the two isothermal processes at $T = T_1$ and $T = T_2$, respectively. The entropy changes of the engine after one cycle is given by

$$\Delta S_{\mathrm{engine}} = \frac{Q_1}{T_1} + \frac{Q_2}{T_2} = nR\ln\frac{V_{\mathrm{b}}}{V_{\mathrm{a}}} + nR\ln\frac{V_{\mathrm{d}}}{V_{\mathrm{c}}}.$$

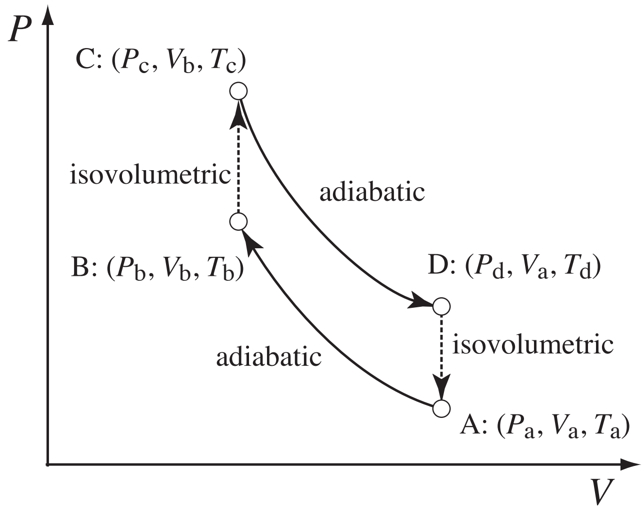
For an ideal gas, equation of the states $PV = nRT$ and adiabatic relation, $PV^{\gamma} = \mathrm{constant}$, leads to $V_{\mathrm{b}}/V_{\mathrm{a}} = V_{\mathrm{c}}/V_{\mathrm{d}}$ and it follows that

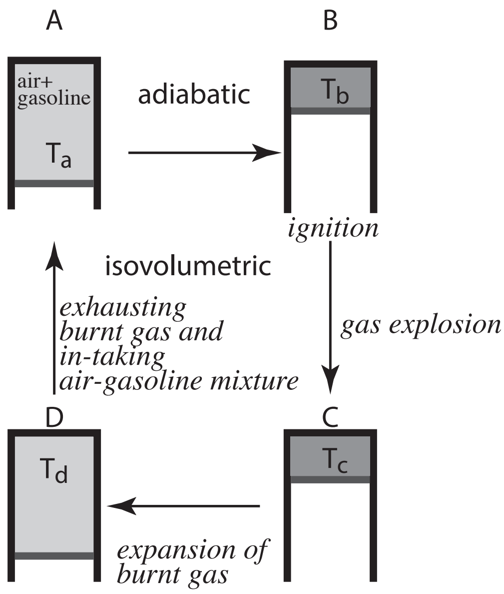
$$\Delta S_{\mathrm{engine}} = nR\left(\ln\frac{V_{\mathrm{b}}}{V_{\mathrm{a}}} + \ln\frac{V_{\mathrm{a}}}{V_{\mathrm{b}}}\right) = 0.$$

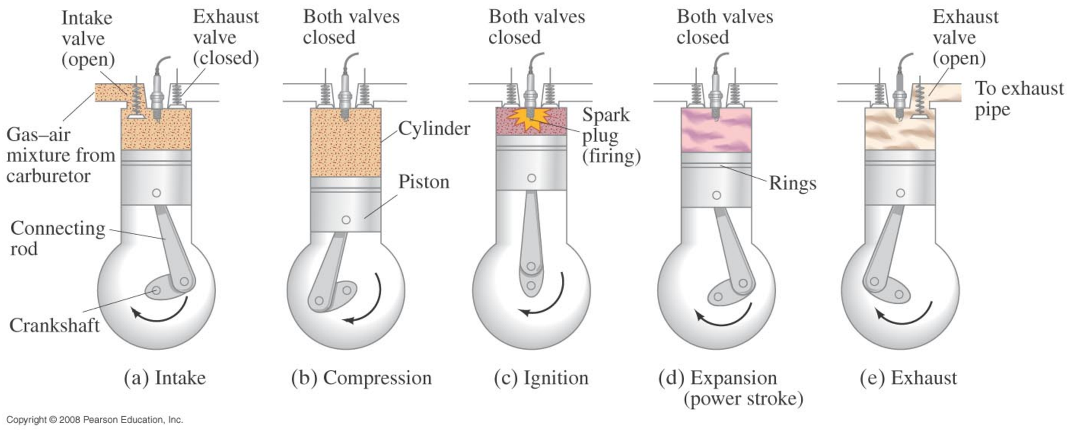
Similarly, the entropy changes of the two heat reservoirs are given by $\Delta S_1 = -Q_1/T_1$ and $\Delta S_2 = -Q_2/T_2$. Thus the total entropy is given by

$$\Delta S = \Delta S_{\mathrm{engine}} + \Delta S_1 + \Delta S_2 = 0.$$

5) For the Otto Cycle shown in the figures below, calculate the efficiency of the Otto cycle engine and compare with that of the Carnot cycle engine, $\varepsilon_{\mathrm{Carnot}} = 1 - T_{\mathrm{a}}/T_{\mathrm{c}}$, where T_{a} and T_{c}, the lowest and highest temperature of the system, respectively. Which one of the two engines is more efficient?







Otto cycle consists of four paths combining the two adiabatic and two isovolumetric paths.

$A(V_a, P_a, T_a)$: Gas (mixture of gasoline with air) in the cylinder

A→B: Adiabatic compression of gas ($Q = 0$, V decreases, P increases, T increases) by the movement of the piston ($W < 0$)

$B(V_b, P_b, T_b)$: Ignition with a spark plug (gasoline) or self-ignition (diesel)

B→C: Q_H generated, P and T increase at the constant volume.

$C(V_b, P_c, T_c)$: Pressure reach at their highest points

C→D: Adiabatic expansion of the gas ($Q = 0$, V increases, P decreases, T decreases) by pushing down the piston ($W > 0$).

$D(V_a, P_d, T_d)$: The volume is at its maximum.

D→A: Q_L to the environment at the constant volume. The burned gas is replaced by the new gas.

The heat in B→C is given by

$$Q_1 = nC_V\left(T_c - T_b\right) > 0$$

and similarly for D→A,

$$Q_2 = nC_V\left(T_a - T_d\right) < 0$$

From the first law of thermo dynamics, $\Delta E_{\text{int}} = Q - W$, the work in one cycle is given by

$$W = Q = Q_1 + Q_2$$

thus the efficiency

$$\varepsilon = \frac{W}{Q_1} = 1 + \frac{Q_2}{Q_1} = 1 + \frac{T_a - T_d}{T_c - T_b}$$

In the adiabatic processes, we have $P_a V_a^\gamma = P_b V_b^\gamma$ and $P_c V_b^\gamma = P_d V_a^\gamma$. Using the ideal gas law,

$$\frac{P_a V_a}{T_a} = \frac{P_b V_b}{T_b} \text{ and } \frac{P_c V_b}{T_c} = \frac{P_d V_a}{T_d}$$

it follows that

$$P_a V_a^\gamma = P_b V_b^\gamma$$

$$P_a V_a^\gamma \frac{T_a}{P_a V_a} = P_b V_b^\gamma \frac{T_b}{P_b V_b}$$

$$T_a V_a^{\gamma-1} = T_b V_b^{\gamma-1}$$

and

$$P_c V_b^\gamma = P_d V_a^\gamma$$

$$P_c V_b^\gamma \frac{T_c}{P_c V_b} = P_d V_a^\gamma \frac{T_d}{P_d V_a}$$

$$T_c V_b^{\gamma-1} = T_d V_a^{\gamma-1}$$

giving

$$T_a - T_d = T_b\left(\frac{V_b}{V_a}\right)^{\gamma-1} - T_c\left(\frac{V_b}{V_a}\right)^{\gamma-1} = \left(\frac{V_b}{V_a}\right)^{\gamma-1}\left(T_b - T_c\right)$$

It follows that

$$\varepsilon = 1 + \frac{T_a - T_d}{T_c - T_b} = 1 + \left(\frac{V_b}{V_a}\right)^{\gamma-1} \frac{T_b - T_c}{T_c - T_b} = 1 - \left(\frac{V_a}{V_b}\right)^{1-\gamma}$$

i.e. the efficiency is a function of the compression ratio V_b/V_a. Since $1 - \gamma < 0$, the efficiency is higher for an engine with a higher compression ratio.

From the adiabatic relation for an ideal gas, $P_a V_a^\gamma = P_b V_b^\gamma$, we obtain $\left(V_a/V_b\right)^\gamma = P_b/P_a$. The equation of the states for an ideal gas, $PV = nRT$, leads to $P_a V_a/T_a = P_b V_b/T_b$, thus we have $P_b/P_a = \left(V_a T_b\right)/\left(V_b T_a\right)$. It follows that

$$\left(\frac{V_a}{V_b}\right)^\gamma = \frac{P_b}{P_a} = \frac{V_a}{V_b}\frac{T_b}{T_a} \text{ thus } \left(\frac{V_a}{V_b}\right)^{1-\gamma} = \frac{T_a}{T_b}.$$

The efficiency can be now written as

$$\varepsilon_{\text{Otto}} = 1 - \frac{T_a}{T_b}$$

By noting that in the isovolumetric process, B→C, we have $P_c > P_b$, thus $T_c = T_b\left(P_c/P_b\right) > T_b$ and $T_a/T_b > T_a/T_c$. It follows that

$$\varepsilon_{\text{Otto}} = 1 - \frac{T_a}{T_b} < 1 - \frac{T_a}{T_c} = \varepsilon_{\text{Carnot}}$$

and the Otto engine is less efficient than the Carnot engine.