General Physics II: Tutorial Material 8

1) In our outer space, the density of matter is 1 atom per cm^3. It is dominated by the hydrogen atom and at a temperature of 2.7 K. What is the rms-speed of those hydrogen atoms? What is the pressure there in the unit of atm?

The mass of the hydrogen atom is $1\ \text{u} = 1.66 \times 10^{-27}\ \text{kg}$, and the number of the hydrogen atom in the universe per m^3 is 10^6. From the Boltzmann's equation, the rms-speed is given by

$$v_{\text{rms}} = \sqrt{3\frac{kT}{m}} = \sqrt{3\frac{1.38 \times 10^{-23}\ \text{J/K} \times 2.7\ \text{K}}{1.66 \times 10^{-27}\ \text{kg}}} = 259\ \text{m/s}$$

The pressure is given by the ideal gas law as

$$P = \frac{NkT}{V} = \frac{10^6 \times 1.38 \times 10^{-23}\ \text{J/K} \times 2.7\ \text{K}}{1\ \text{m}^3} = 3.726 \times 10^{-17}\ \text{Pa}$$

Using the conversion factor, 1 atm $= 1.01 \times 10^5\ \text{Pa}$, we obtain

$$P = 3.726 \times 10^{-17}\ \text{Pa} = \frac{3.726 \times 10^{-17}}{1.01 \times 10^5} = 3.7 \times 10^{-22}\ \text{atm}$$

2) The lowest pressure attainable using the best available vacuum technique is about $10^{-12}\ \text{Nm}^{-2}$. At such a pressure, how many molecules are there per cm^3 at 0° C?

Assuming the ideal gas law is valid, the number of molecules is given by

$$N = \frac{PV}{kT}$$

where k is the Boltzmann constant, $= 1.38 \times 10^{-23}\ \text{J/K} = 1.38 \times 10^{-23}\ \text{Nm/K}$. For $P = 10^{-12}\ \text{Nm}^{-2}$, $V = 1\ \text{cm}^3 = 10^{-6}\ \text{m}^3$, and T = 273 K, we obtain the number of molecule per cubic cm to be

$$N = \frac{10^{-12}\ \text{Nm}^{-2} \times 10^{-6}\ \text{m}^3}{1.38 \times 10^{-23}\ \text{NmK}^{-1} \times 273\ \text{K}} = 265$$

3) Show that the rms-speed of gas molecules is given by $v_{\text{rms}} = \sqrt{3P/\rho}$, where P and ρ are the pressure and density of the gas respectively.

From the Maxwell's distribution, the rms-velocity is given by

$$v_{\text{rms}} = \sqrt{3\frac{kT}{m}}$$

where k is the Boltzmann's constant and m is the mass of the gas molecule. The ideal gas law in terms of the Boltzmann's constant is given by $PV = NkT$ where N is the number of molecule. It follows that

$$\frac{kT}{m} = \frac{PV}{Nm} = \frac{P}{Nm/V} = \frac{P}{\rho}$$

where $\rho \equiv Nm/V$ is the density of the gas. It follows that

$$v_{\text{rms}} = \sqrt{3\frac{kT}{m}} = \sqrt{3\frac{P}{\rho}}$$

4) How many joules and kilocalories are generated when the breaks are used to stop the car running at 95 km/h and weight 1200 kg.

The kinetic energy of the car is given by

$$E_{\text{kinetic}} = \frac{mv^2}{2} = \frac{1200\text{ kg}}{2}(95\text{ km/h})^2 = \frac{1200\text{ kg}}{2}\left(\frac{95000}{3600}\frac{\text{m}}{\text{s}}\right)^2 = 4.18\times10^5\text{ J}$$

From the conversion factor, 1 cal = 4.186 J, this corresponds to

$$\frac{4.18\times10^5}{4.186} \approx 1\times10^2\text{ kcal}$$

This amount of energy will be generated when the breaks are used to stop the car.

5) An ideal gas is kept in a container with rigid walls. How can we reduce the pressure of the gas? How much work the gas will do during that process?

Seeing from the ideal gas law,

$$P = \frac{nRT}{V}$$

the pressure can be reduced by lowing the temperature of the gas. Since the gas is in a container with rigid walls, the volume does not change. Therefore no work is done.

6) There are N indistinguishable gas molecules uniformly distributed in a box with a volume V. Consider a small region in the box with a volume V_1.

a) What is the probability to find any one but only one molecule in this region?

b) What is the probability to find any n molecules in this region?

c) What is the average number of molecules, $\langle n \rangle$, and its standard deviation $\Delta n \equiv \sqrt{\langle n^2 \rangle - \langle n \rangle^2}$, where $\langle n^2 \rangle$ is the average of n^2, in this region?

d) If N is of the order of the Avogadro number, i.e. about 10^{24}, and the volume of the considered region is about 1% of the total volume, how large is $\Delta n / \langle n \rangle$? What does it mean?

NB: Following formula might be useful:

$$(p+q)^M = \sum_{m=0}^{M} \frac{M!}{m!\,(M-m)!} p^m q^{M-m}$$

$$\sum_{m=0}^{M} m \frac{M!}{m!\,(M-m)!} p^m q^{M-m} = Mp(p+q)^{M-1}$$

$$\sum_{m=0}^{M} m^2 \frac{M!}{m!\,(M-m)!} p^m q^{M-m} = Mp(p+q)^{M-1} + M(M-1)p^2(p+q)^{M-2}$$

a) Since molecules are uniformly distributed in a volume V, the probability to find **one particular** molecule in a volume V_1, p, must be proportional to V_1. When $V_1 = V$, this probability must be p=1. By combining the two facts, we get

$$p = \frac{V_1}{V}$$

and equally for the probability to find it outside of the volume V_1, q, is

$$q = \frac{V - V_1}{V} = 1 - \frac{V_1}{V} = 1 - p$$

thus

$$p + q = 1,$$

i.e. the probability to find one particular molecule inside or outside of the volume is one, which is logical. The probability for out of N molecules to have **one particular molecule** in the volume and **all the other $N-1$ molecules** outside of the volume is given by

$$pq^{N-1} = \left(\frac{V_1}{V}\right)\left(1 - \frac{V_1}{V}\right)^{N-1}$$

The probability for out of N molecules to have **any one but only one** molecule in the volume, $P(1)$, is then given by

$$P(1) = N\left(\frac{V_1}{V}\right)\left(1 - \frac{V_1}{V}\right)^{N-1}$$

since the one particle in the volume can be any of the N molecules.

b) By generalising 1) and using the number of combinations to select n molecules out of N molecules, the probability to find n indistinguishable molecules in the region, $P(n)$, is given by

$$P(n) = \frac{N!}{n!\,(N-n)!}\left(\frac{V_1}{V}\right)^n\left(1 - \frac{V_1}{V}\right)^{N-n}$$

c) The average, $\langle n \rangle$, and squared average, $\langle n^2 \rangle$, are given by

$$\langle n \rangle = \sum_{n=0}^{N} nP(n) = \sum_{n=0}^{N} n \frac{N!}{n!\,(N-n)!} \left(\frac{V_1}{V}\right)^n \left(1 - \frac{V_1}{V}\right)^{N-n} = N\left(\frac{V_1}{V}\right)$$

and

$$\langle n^2 \rangle = \sum_{n=0}^{N} n^2 P(n) = \sum_{n=0}^{N} n^2 \frac{N!}{n!\,(N-n)!} \left(\frac{V_1}{V}\right)^n \left(1 - \frac{V_1}{V}\right)^{N-n} = N\left(\frac{V_1}{V}\right) + N(N-1)\left(\frac{V_1}{V}\right)^2$$

respectively. It follows that

$$\begin{aligned}\langle n^2 \rangle - \langle n \rangle^2 &= N\left(\frac{V_1}{V}\right) + N(N-1)\left(\frac{V_1}{V}\right)^2 - N^2\left(\frac{V_1}{V}\right)^2 \\ &= N\left(\frac{V_1}{V}\right)\left(1 - \frac{V_1}{V}\right)\end{aligned}$$

and the standard deviation is given by

$$\Delta n = \sqrt{N\left(\frac{V_1}{V}\right)\left(1 - \frac{V_1}{V}\right)}$$

d) The ratio, $\Delta n / \langle n \rangle$, is given by

$$\frac{\Delta n}{n} = \frac{1}{\sqrt{N}} \sqrt{\frac{V}{V_1} - 1}$$

and for $N \approx 10^{24}$ and $V/V_1 \approx 10^{-2}$, we have

$$\frac{\Delta n}{n} \approx 10^{-11}$$

i.e. the relative statistical fluctuations of the number of molecules from the average number is very small.