General Physics II: Tutorial Material 7

1) A helium-filled balloon escapes a child's hand at sea level where the atmosphere is in 1 atm and 20.0° C. When it reaches an altitude of 3600 m where the temperature is 5.0° C and the pressure 0.68 atm, how will its volume compare to that at the sea level.

We consider He gas to be an ideal gas. Since the He in the balloon does not escape, the volume, pressure and temperature at the sea level (0 m) and those at 3600 m follows the relation

$$\frac{P_0 V_0}{T_0} = \frac{P_{3600} V_{3600}}{T_{3600}}$$

thus

$$\frac{V_{3600}}{V_0} = \frac{P_0 T_{3600}}{P_{3600} T_0}$$

By inserting P_0 = 1 atm, P_{3600} = 0.68 atm, T_0 = 273+20 K, and T_{3600} = 273+5 K, we obtain

$$\frac{V_{3600}}{V_0} = \frac{1 \times 278}{0.68 \times 293} = 1.4$$

i.e. the volume increases by 40% at 3600 m.

2) An air bubble with a diameter of 3.60 mm was created at the bottom of the lake, which is $d_{\text{watter}} = 2.5\text{ m}$ deep. When the bubble reached the surface of the lake, where the temperature is 27° C, the diameter of the air bubble became 4.00 mm. The pressure of the atmosphere at the surface of the lake was 1 atm. What is the temperature of the water at the bottom of the lake? The density of the water is $\rho_W = 1 \times 10^3\ \text{kg/m}^3$ and the gravitational acceleration constant $g = 9.80\,\text{m/s}^2$, and assume that the air behaves as an ideal gas.

The pressure, volume and temperature of the bubble at the surface (bottom) of the lake are denoted as P_s (P_b), V_s (V_b) and T_s (T_b), respectively. Since the amount of air in the bubble did not change, $P_s V_s / T_s = nR = P_b V_b / T_b$, thus

$$T_b = \frac{P_b V_b}{P_s V_s} T_s$$

At the surface, we have

$$P_s = 1\text{ atm} = 1.013 \times 10^5\text{ Pa} = 1.013 \times 10^5\ \text{N/m}^2$$

$$V_s = 4\pi r^3/3 = 4\pi \left(4 \times 10^{-3}\ \text{m}/2\right)^3 / 3 = 3.351 \times 10^{-8}\ \text{m}^3\ ,$$

$$T_s = 27^\circ\ \text{C} = 300.2^\circ\ \text{K}$$

The pressure at the bottom of the lake is given by $P_b = P_s + P_{\text{water}}$ where P_{water} is the pressure of the 2.5 m of water due to the gravitational force given by

$$P_{\text{water}} = g \times \rho_{\text{water}} \times d_{\text{water}} = \left(9.80\ \text{m/s}^2\right) \times \left(1 \times 10^3\ \text{kg/m}^3\right) \times \left(2.5\ \text{m}\right) = 2.45 \times 10^4\ \text{N/m}^2$$

It follows that

$$P_{\text{b}} = 1.258 \times 10^5\ \text{N/m}^2$$

$$V_{\text{b}} = 4\pi r^3/3 = 4\pi\left(3.6 \times 10^{-3}\ \text{m}/2\right)^3/3 = 2.442 \times 10^{-8}\ \text{m}^3$$

thus

$$T_{\text{b}} = \frac{P_{\text{b}} V_b}{P_{\text{s}} V_{\text{s}}} T_{\text{s}} = 272^\circ\ \text{K}$$

3) The rms speed of molecules in a gas at 20.0° C is to be increased by 2.0%. To what temperature must it be raised?

The rms speed of molecule in a gas is given by

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

The ratio at two different temperature is then given by

$$\frac{v'_{\text{rms}}}{v_{\text{rms}}} = \sqrt{\frac{3kT'}{m}} \bigg/ \sqrt{\frac{3kT}{m}} = \sqrt{\frac{T'}{T}}$$

We have T = 20+273 = 293 K and $v'_{\text{rms}}/v_{\text{rms}} = 1.02$, makes

$$T' = \left(\frac{v'_{\text{rms}}}{v_{\text{rms}}}\right)^2 T = 1.02^2 \times 293 = 305\ \text{K}$$

i.e. the temperature must be increased by 12 ˚C.

4) If you double the mass of the molecules in a gas, is it possible to change the temperature to keep the velocity distribution from changing? If so, how much change do you need to make to the temperature?

The velocity distribution is given by the Maxwell distribution

$$F(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{m}{2kT}v^2\right)$$

In order to keep the velocity distribution unchanged, *m*/*T* must remain the same. Therefore, when the mass is doubled, the temperature of the gas must be doubled to keep the same velocity distribution.

5) There are four coins with two faces, head and tail. Each coin has 50% probability to show head and 50% probability to show tail, when tossed individually. When we toss the four coins together:

a) How many head-tail configurations are there if we can distinguish individual coins? What are the probabilities for those configurations?

b) How many head-tail configurations are there if we cannot distinguish individual coins? Which configuration has the highest probability to be realised?

a) Possible number of configurations for the case a) are

All coins are head	1 configuration
Three coins are head and one con tail	4 configurations
Two coins are head and two coins tail	6 configurations
One coin is head and three coins tail	4 configurations
All coins are tail	1 configuration

i.e. 16 configurations and each configuration as a probability of $0.5^4 = 0.0625$.

b) there are only 5 configurations

All coins are head

Three coins are head and one con tail

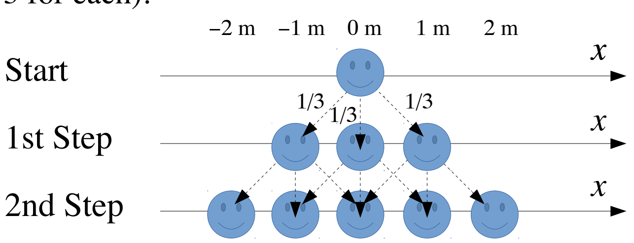
Two coins are head and two coins tail

One coin is head and three coins tail

All coins are tail

and the third configuration, i.e. two heads and two tails, has the highest probability of $0.0625 \times 6 = 0.375$.

6) A drunken person is standing at $x = 0$ m. When the drunken person makes one step, the person may go to the left (negative direction in x by 1m), remain at the same position or to the right (positive direction in x by 1 m) with a same probability (1/3 for each).



a) What is the probability for the drunken person to be at $x = -8$ m, -7 m, -6 m, -5 m, -4 m, -3 m, -2 m, -1 m, -0 m, 1 m, 2 m, 3 m, 4 m, 5 m, 6 m, 7 m and 8 m after 1, 2, 4 and 7 steps?

b) What are the mean value, $\langle x \rangle$, and rms, x_{rms}, for x after 1, 2, 4 and 7 steps?

Plots below give four probability distributions in x following the Gauss distribution, $G(x)$, given by

$$G(x) = \frac{1}{\sqrt{2\pi x_{\text{rms}}^2}} \exp\left[-\frac{\left(x - \langle x \rangle\right)^2}{2x_{\text{rms}}^2} \right]$$

with x_{rms} equal to those obtained for 1, 2, 4 and 7 steps above, but not necessarily in this order.

c) Find out which Gauss distribution belong to which steps.

d) Superimpose the probability distributions of the x position of the drunken person on the Gauss distribution of corresponding steps. What kind of conclusion can you draw from comparing the distributions?

a) Since the drunken person moves to the left and right with a same probability, the probability distribution in x is symmetric at around $x = 0$ m. After n-steps, the farthest the person can reach is $x = \pm n$ m with a probability $(1/3)^n$, and there are in total $2n+1$ x positions with non-zero probabilities. Further more, there are, in total, 3^n different paths the person can take. Now we consider two approaches to obtain the solutions:

Step by step approach

1) After 1st step, the person arrives at $x = -1$ m, 0 m or 1 m with 1/3, 1/3 and 1/3 probabilities respectively.
2) After 2nd step, the person can reach $x = -2$ m from $x = -1$ m of the 1st step with a probability of $1/3 \times 1/3 = 1/9$. The position, $x = -1$ m, after the 2nd step can be reached from $x = -1$ m and 0 m of the first step with probabilities, 1/9 and 1/9, respectively, thus the total probability is 2/9. The $x = 0$ m after the 2nd step can be reached from three positions in the 1st step, $x = -1$ m, 0 m or 1 m, with probabilities 1/9, 1/9 and 1/9, respectively, thus the total probability is 1/3. For the remaining positions, $x = 1$ m and 2 m, the symmetry around $x = 0$ m can be used.
3) After 3rd step, the person can reach $x = -3$ m from $x = -2$ m of the 2nd step with a probability of $1/9 \times 1/3 = 1/27$. The position, $x = -2$ m, after the 3rd step can be reached from $x = -2$ m and -1 m of the first step with probabilities, 1/27 and $2/9 \times 1/3 = 2/27$, respectively, thus the total probability is 1/9. The $x = -1$ m after the 3rd step can be reached from the three positions in the 2nd step, $x = -2$ m, -1 m or 0 m, with probabilities 1/27, $2/9 \times 1/3 = 2/27$ and $1/3 \times 1/3 = 1/9$, respectively, thus the total probability is 2/9. Finally, the $x = 0$ m after the 3rd step can be reached from the three positions in the 2nd step, $x = -1$ m, 0 m or 1 m, with probabilities $2/9 \times 1/3 = 2/27$, $1/3 \times 1/3 = 1/9$ and $2/9 \times 1/3 = 2/27$ respectively, thus the total probability is 7/27. For the remaining positions, the symmetry around $x = 0$ m can be used.
4) Continue for the further steps.

More general approach

The total number of the step, n, can be written as

$$n = n_L + n_R + n_C$$

where n_L, n_R and n_C are the numbers of steps moving to the left, right and remaining at the place, respectively. By denoting the position of the person after n steps to be $x = -m$ m, m is given by

$$m = n_L - n_R$$

From the two equations, we obtain

$$n = m + 2n_R + n_C$$

i) If $m = n$, i.e. arriving at the furthest negative position in x, from the equation above, it follows that

$$2n_R + n_C = 0$$

leading to

$$n_R = 0,\ n_C = 0,\ n_L = n$$

The total number of paths to this point is given by a trinomial combination

$$N^n_{n_R n_C} \equiv \frac{n!}{(n - n_R - n_C)!n_R!n_C!}$$

as

$$N^n_{00} = \frac{n!}{n!0!0!} = 1$$

and the probability for this to happen by

$$\frac{N^n_{00}}{3^n} = \left(\frac{1}{3}\right)^n$$

ii) If $m = n - 1$, i.e. arriving at the position $x = -(n-1)$ m after n-steps, we obtain

$$2n_R + n_C = 1$$

which leads to

$$n_R = 0,\ n_C = 1,\ n_L = n - 1$$

and the number of possible paths to reach this point is

$$N^n_{01} = \frac{n!}{(n-1)!0!1!} = n$$

The probability to arrive here after n-steps is then

$$\frac{N^n_{01}}{3^n} = \frac{n}{3^n}$$

iii) If $m = n - 2$, i.e. arriving at the position $x = -(n-2)$ m after n-steps, we obtain

$$2n_R + n_C = 2$$

which leads to

$$n_R = 0,\ n_C = 2,\ n_L = n - 2 \text{ or } n_R = 1,\ n_C = 0,\ n_L = n - 1$$

and the number of possible paths to reach this point is

$$N^n_{02} + N^n_{10} = \frac{n!}{(n-2)!0!2!} + \frac{n!}{(n-1)!1!0!}$$

The probability to arrive here after n-steps is

$$\frac{N^n_{02} + N^n_{10}}{3^n}$$

iv) For $m = n - 3$,

$$2n_R + n_C = 3$$

which leads to

$$n_R = 0,\ n_C = 3,\ n_L = n - 3 \text{ or } n_R = 1,\ n_C = 1,\ n_L = n - 2$$

and the number of possible paths to reach this point is

$$N^n_{03} + N^n_{11} = \frac{n!}{(n-3)!0!3!} + \frac{n!}{(n-2)!1!1!}$$

The probability to arrive here after n-steps is

$$\frac{N^n_{03} + N^n_{11}}{3^n}$$

v) Continue till $m = 0$, and the rest can be obtained by the symmetry at around $x = 0$.

To check, we calculate the case for $n = 3$ and $m = 0$, i.e. the person arrives $x = 0$ after taking three steps, which corresponds to iv) giving the probability to be

$$\frac{N_{03}^3 + N_{11}^3}{3^3} = \frac{1+6}{27} = \frac{7}{27}$$

agreeing with the result given in 3) above.

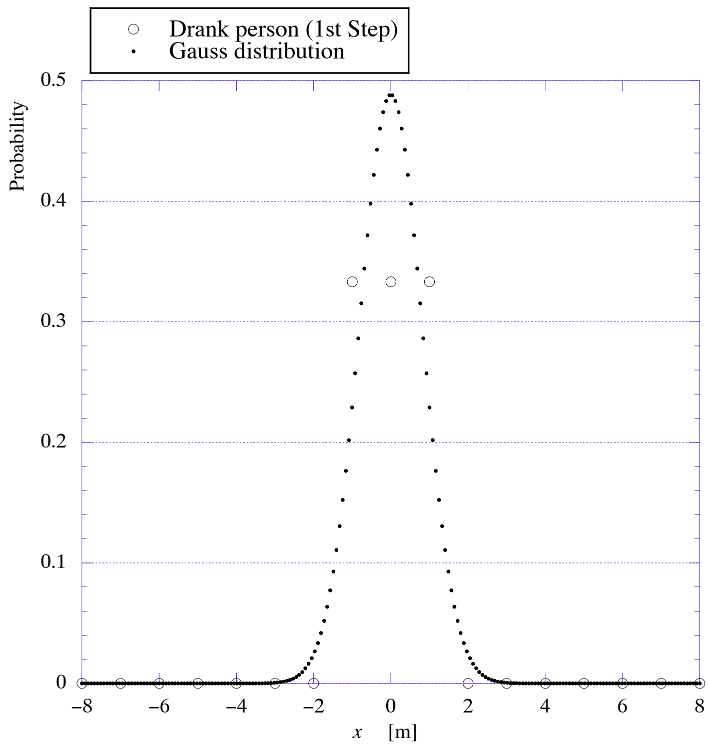
Table below shows the probabilities for the drunken person to be at different x positions after 1, 2, 3, 4 and 7 steps which can be obtained in two ways described above.

x [m]	Step 1	Step 2	Step 3	Step4	Step 7
−8	0	0	0	0	0
−7	0	0	0	0	1/2187
−6	0	0	0	0	7/2187
−5	0	0	0	0	28/2187
−4	0	0	0	1/81	77/2187
−3	0	0	1/27	4/81	161/2187
−2	0	1/9	1/9	10/81	266/2187
−1	1/3	2/9	2/9	16/81	119/729
0	1/3	1/3	7/27	19/81	131/729
1	1/3	2/9	2/9	16/81	119/729
2	0	1/9	1/9	10/81	266/2187
3	0	0	1/27	4/81	161/2187
4	0	0	0	1/81	77/2187
5	0	0	0	0	28/2187
6	0	0	0	0	7/2187
7	0	0	0	0	1/2187
8	0	0	0	0	0

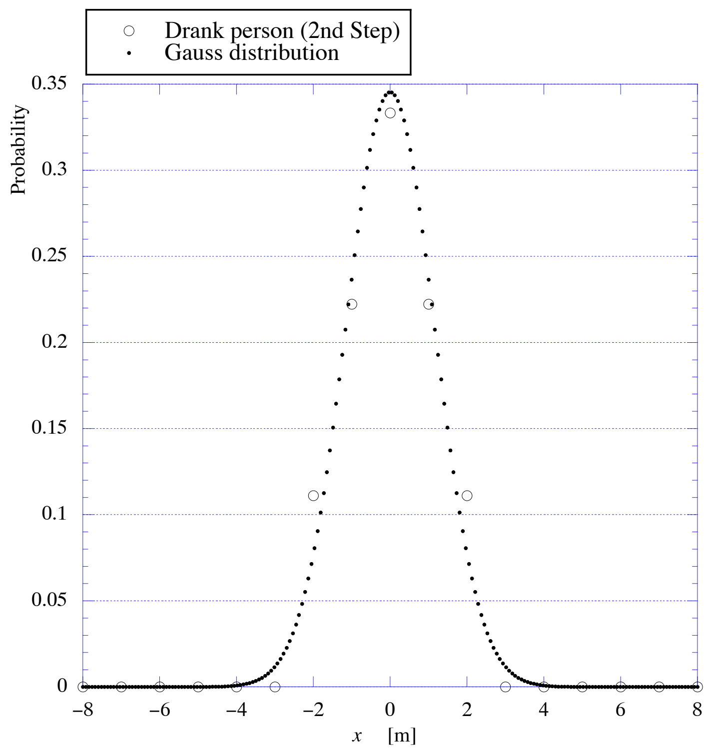
b) The mean position and rms in x, $\langle x \rangle$ and x_{rms} respectively, are given by

	Step1	Step2	Step 4	Step 7
$\langle x \rangle$ [m]	0	0	0	0
x_{rms} [m]	0.8165	1.1547	1.6330	2.1602

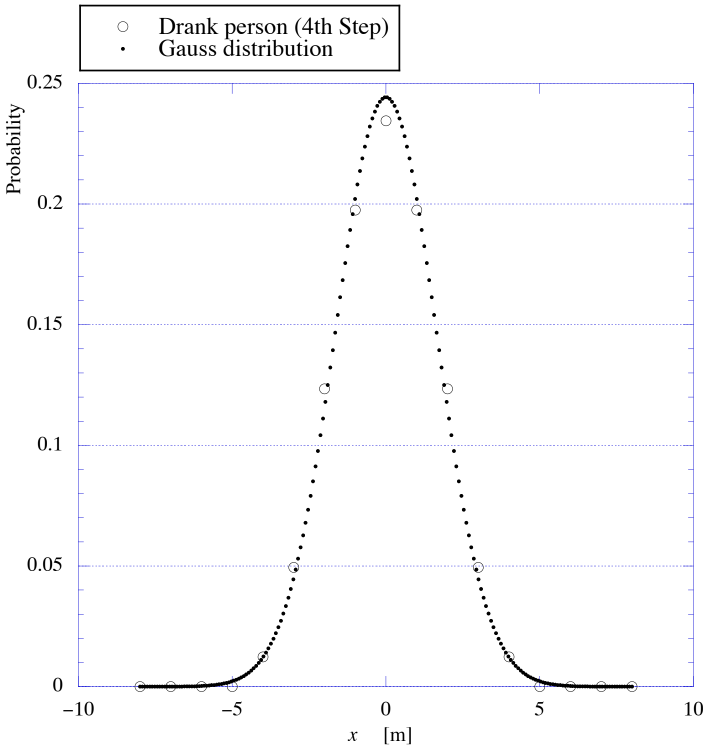
c,d) Finally, figures in the next page shows the probabilities given in the table compared with the Gauss distribution taking $\langle x \rangle$ and x_{rms} as the mean and standard deviation for 1, 2, 4 and 7 steps. After 2 steps, the Gauss distribution already describes the probability distribution of the position of the drunken person rather well and the agreement gets better with increasing number of steps. This is a demonstration that by repeating a random process, which itself does not follow the Gauss distribution, resulting in a distribution follows the Gauss distribution. This is why in physics distributions of variables resulting from random processes are often taken as Gaussian form.

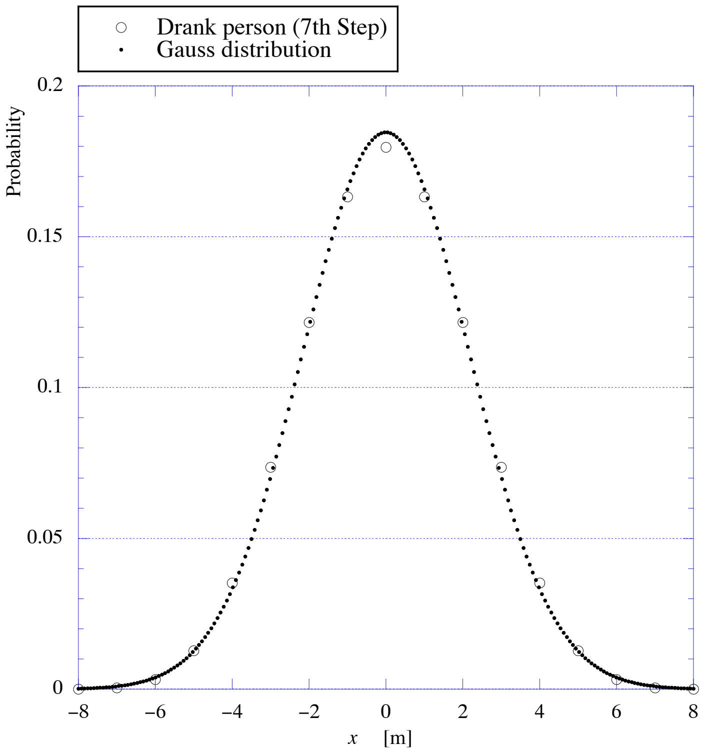


Drank person (1st Step)
Gauss distribution
Probability
0.5
0.4
0.3
0.2
0.1
0
−8
−6
−4
−2
0
2
4
6
8
x [m]



Drank person (2nd Step)
Gauss distribution
Probability
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−8
−6
−4
−2
0
2
4
6
8
x [m]


Drank person (4th Step)
Gauss distribution
Probability
0.25
0.2
0.15
0.1
0.05
0
−10
−5
0
5
10
x [m]


Drank person (7th Step)
Gauss distribution
Probability
0.2
0.15
0.1
0.05
0
−8
−6
−4
−2
0
2
4
6
8
x [m]