General Physics II at EPFL

(2018-2019 WS, Wed 17:15-19:00 and Thu 8:15-10:00, Exercise Thu 10:15-12:00)

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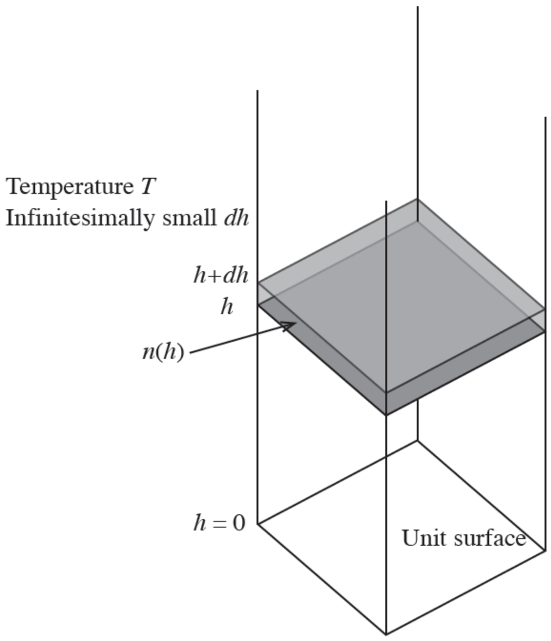
Mock Examination II

Date: Friday 15 May 2019, 17:15-19:00
Room: CE1
3 Problems
Thermodynamics

Problem 1

Boltzmann factor gives that the number of gas molecules with an energy E, for a gas at an absolute temperature T to be $\propto e^{-E/kT}$, where k is the Boltzmann constant. A gas molecule on the earth gains potential energy mgh, where h is the altitude of the molecule position measured from the sea level, m the mass of the gas molecule and g gravitational constant. By assuming that the energy of the gas molecule is totally given by the earth gravity (a reasonable assumption since the earth atmosphere does not escape to the outer space) and the temperature does not depend on the altitude:

1) Obtain the number of gas molecules for a unit surface with an infinitesimally small thickness at an altitude of h, $n(h)$, using n_0, which is the number of gas molecules per unit surface with an infinitesimally small thickness at $h = 0$.
2) Assuming that the gas is an ideal gas, obtain the pressure of the gas, $P(h)$, at an altitude h from $n(h)$.
3) Show that the gas pressure at the sea level is equal to the gravity force acting on the total mass of the gas molecules (altitudes from 0 to ∞) per unit surface.



Solutions:

1) By combining the Boltzmann factor, $e^{-E/kT}$, energy of a gas molecule at an altitude h, mgh, and knowing that $n(0) = n_0$, $n(h)$ is given by

$$n(h) = n_0 e^{-mgh/kT}$$

2) Since the gas follows the ideal gas law, $PV = nkT$, and n/V is the particle density, P for an infinitesimally thin slice of the gas volume at a height, h, can be given as

$$P(h) = \left(\frac{n}{V} \text{ at } h\right)kT = n(h)kT = n_0 kTe^{-mgh/kT}$$

3) From 1), P at the sea level, $h = 0$, is given by

$$P(0) = n_0 kT$$

The total umber of gas molecules, N, contained in an infinitely high column with a unit surface bottom is given by

$$N = \int_0^\infty n(h)\,dh = n_0 \int_0^\infty e^{-mgh/kT}\,dh = \frac{n_0 kT}{mg}$$

and the total mass for those molecules, M, is given by

$$M = mN = \frac{kTn_0}{g}$$

thus $n_0 kT = Mg$, leading to

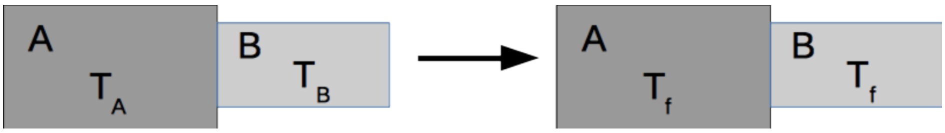
$$P(0) = n_0 kT = Mg\ ,$$

i.e. the pressure at the sea level is equal to the gravitational force acting on the mass of the gas molecule per unit surface.

Problem 2

We consider a system of two solid bodies, A and B, with their masses, m_A and m_B, and specific heats per mass, c_A and c_B, respectively. And their initial absolute temperatures are T_A and T_B. Now the two bodies are put together under thermal contact while the system is thermally isolated from the environment.

1) Using the 1st law of thermodynamics, calculate the temperature T_f, when the system has reached thermal equilibrium.
2) Show that T_f is always in between T_A and T_B.
3) Calculate the entropy changes for A and B, ΔS_A and ΔS_B, between the initial and the final thermal equilibrium states.
4) From ΔS_A and ΔS_B obtained above, show that the total entropy change, $\Delta S_A + \Delta S_B$, is always ≥ 0.



Solutions

1) Heats for A and B for reaching thermal equilibrium are given by $Q_A = m_A c_A (T_f - T_A)$ and $Q_B = m_B c_B (T_f - T_B)$, respectively. There is no heat from the outside since the system is thermally isolated, and there is no work, thus $Q_A + Q_B = 0$, i.e. $m_A c_A (T_f - T_A) + m_B c_B (T_f - T_B)$=0. This leads to

$$T_f = \frac{m_A c_A T_A + m_B c_B T_B}{m_A c_A + m_B c_B}$$

2) By introducing $T_A - T_B = \delta$, T_f can be written as

$$T_f = T_A - \frac{m_B c_B \delta}{m_A c_A + m_B c_B} = T_B + \frac{m_A c_A \delta}{m_A c_A + m_B c_B}$$

where clearly

$$0 < \frac{m_B c_B}{m_A c_A + m_B c_B} < 1 \text{ and } 0 < \frac{m_A c_A}{m_A c_A + m_B c_B} < 1$$

Thus, if $T_A - T_B = \delta > 0$, $T_f < T_A$ and $T_f > T_B$, and if $T_A - T_B = \delta < 0$, $T_f > T_A$ and $T_f < T_B$, i.e. T_f is always in between T_A and T_B.

3) The process can be reversed by introducing another solid bodies with appropriate temperatures to restore the original temperatures for A and B. Therefore, the entropy change can be calculated as $\Delta S = \int dQ/T$, i.e.

$$\Delta S_A = \int_{T_A}^{T_f} \frac{m_A c_A dT}{T} = m_A c_A \ln \frac{T_f}{T_A} \text{ and } \Delta S_B = \int_{T_B}^{T_f} \frac{m_B c_B dT}{T} = m_B c_B \ln \frac{T_f}{T_B}$$

4) The total change of entropy is given by $\Delta S_{tot} = \Delta S_A + \Delta S_B$. By introducing $x = T_B/T_A$, where the range of x is from 0 to ∞,

$$T_f = \frac{m_A c_A + m_B c_B x}{m_A c_A + m_B c_B} T_A = \frac{m_A c_A + m_B c_B x}{(m_A c_A + m_B c_B) x} T_B$$

thus

$$\frac{T_f}{T_A} = \frac{m_A c_A + m_B c_B x}{m_A c_A + m_B c_B} \text{ and } \frac{T_f}{T_B} = \frac{m_A c_A + m_B c_B x}{\left(m_A c_A + m_B c_B\right) x}$$

and ΔS_{tot} can be written as

$$\Delta S_{tot} = \left(m_A c_A + m_B c_B\right) \ln \frac{m_A c_A + m_B c_B x}{m_A c_A + m_B c_B} - m_B c_B \ln x$$

at $x = 1$, i.e. $T_A = T_B$, $\Delta S_{tot} = 0$ that makes sense since if there is no temperature difference between A and B, they are already in thermal equilibrium and nothing happens. By taking derivative of ΔS_{tot} respect to x, we obtain

$$\frac{d\Delta S_{tot}}{dx} = \frac{m_A c_A m_B c_B \left(x-1\right)}{\left(m_A c_A + m_B c_B x\right) x}$$

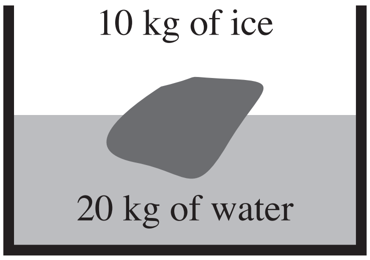
leading to $d\Delta S_{tot}/dx < 0$ for x<1 and $d\Delta S_{tot}/dx > 0$ for x>1, i.e. from x=0 to 1 ΔS_{tot} is monotonically decreasing and reach to 0 at x=1, then monotonically increasing from x=1 to ∞. Therefore $\Delta S_{tot} \geq 0$.

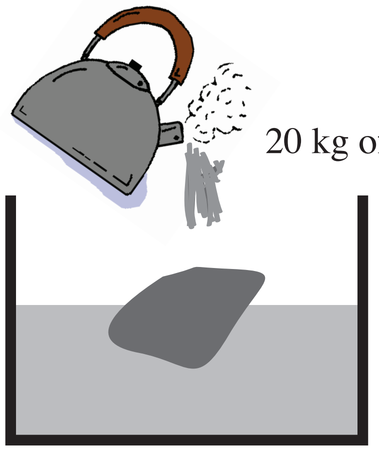
Problem 3

Consider an isolated system with a 10 kg ice block in 20 kg of water, which is in a thermal equilibrium state. In the following, assume that the specific heat and heat of fusion of the ice are $0.5\ \text{kcal}/(\text{kg}\cdot\text{C})$ and $80\ \text{kcal}/\text{kg}$, respectively, and specific heat of the water is $1\ \text{kcal}/(\text{kg}\cdot\text{C})$.

1) What is the temperature of the system?

2) If we add 20 kg of water at 90° C to the system, what will be the temperature of the system after reaching its equilibrium and what are the constituents of the system?





Solutions:

1) Since the water and ice are in thermal contact and in equilibrium, their temperatures are identical. The ice melts above 0° C and the water freezes below 0° C. Since they coexist, the temperature of the system must be at 0° C.

2) In order to melt the entire ice,

$$Q_{\text{ice}} = 80\ \frac{\text{kcal}}{\text{kg}} \times 10\ \text{kg} = 800\ \text{kcal}$$

of thermal energy is needed. If 20 kg of water at $90^\circ\ C$ is cooled down to 0° C,

$$Q_{90} = 1\ \frac{\text{kcal}}{\text{kg}\cdot\text{C}} \times 20\ \text{kg} \times (90-0)^\circ\ \text{C} = 1800\ \text{kcal}$$

is released, which is more than the thermal energy needed to melt the ice. Therefore, the system after reaching the thermal equilibrium consists of water only. For the 20 kg of water at $90^\circ\ C$ to provide the necessary thermal energy to melt the ice, its temperature should go down by

$$\frac{Q_{\text{ice}}}{1\ \frac{\text{kcal}}{\text{kg}\cdot\text{C}} \times 20\ \text{kg}} = \left(\frac{800}{20}\right)^\circ \text{C} = 40^\circ\ \text{C}$$

i.e. to 50° C. Now we have 30 kg of water at 0° C and 20 kg of water at 50° C. By denoting T to be final temperature by mixing the two,

$$1\ \frac{\text{kcal}}{\text{kg}\cdot\text{C}} \times 30\ \text{kg} \times T^\circ\ \text{C} = 1\ \frac{\text{kcal}}{\text{kg}\cdot\text{C}} \times 20\ \text{kg} \times (50-T)^\circ\ \text{C}$$

leading to $T = 20^\circ$ C . In conclusion, the equilibrium state is 50 kg of water at 20° C .