Problem 1

An observer on the earth sees that two rockets, Rocket-1 and Rocket-2, are departing from the earth in opposite directions to each other with the same speed, which is 50% of the speed of light in vacuum, i.e. $0.5c$. What is the speed of Rocket-2 seen by an observer in Rocket-1? Rocket-1 emits light toward Rocket-2, sometime after the departure. Can the light reach Rocket-2?

Solution

We introduce two inertial frames, S and S', where the both x-axes are aligned with the direction of the Rockets flying. In the S frame, Rocket 1 is at rest and in S', the Earth is at rest. The relative speed between the two rockets is then given by the speed of Rocket 2 in the S frame. The direction of the x-axes is chosen so that Rocket 2 flies toward the positive direction in x. In the S' frame, the velocity of the Rocket 2 is given by $u'_x = 0.5c$, $u'_y = 0$ and $u'_z = 0$, and the frame S' is moving with a velocity $v = 0.5c$ in the positive x direction respect to the frame S. The velocity transformation formula relate the velocity of the Rocket 2 in S' to that in S as

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2},\ u_y = \frac{u'_y\sqrt{1-(v/c)^2}}{1 + vu'_x/c^2},\ u_z = \frac{u'_z\sqrt{1-(v/c)^2}}{1 + vu'_x/c^2}$$

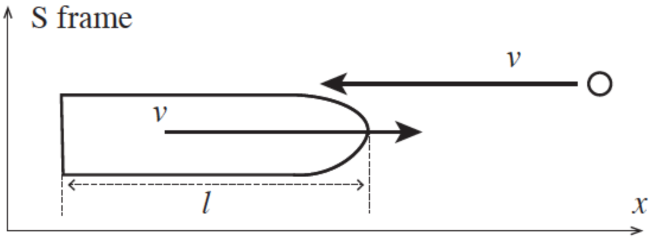
It follows that

$$u_x = \frac{0.5c + 0.5c}{1 + 0.5c \times 0.5c/c^2} = \frac{1.0}{1.25}c,\ u_y = 0,\ u_z = 0$$

i.e. the relative speed between Rocket 1 and Rocket 2 is less than the speed of the light. Therefore the light sent by Rocket 1 will arrive at Rocket 2.

Problem 2

We chose inertial S frame to be the one where the earth is at rest. In S frame, a rocket with a proper length of l_0 is moving in positive x direction with a velocity v and a meteorite in a negative x direction with a velocity v. As shown in the figure, the rocket and meteorite are crossing each other and the meteorite travels through the full length of the rocket. Determine the time it takes for the meteorite to travel through the rocket in S, S' and S'' frame, ΔT, $\Delta T'$ and $\Delta T''$, where in S' the rocket is at rest and in S'' the meteorite is at rest.



Solution

With respect to S frame, S' and S'' are moving with a velocity v in the positive and negative x direction, respectively. In S' frame, the velocity of the meteorite is given with the formula for the addition of velocity as

$$v'_m = \frac{2v}{1+(v/c)^2}$$

Similarly, the velocity of the rocket in S'' frame is given by

$$v''_r = \frac{2v}{1+(v/c)^2}$$

Since the rocket is at rest in S', the length of the rocket in S' is the proper length, l_0. In S and S'' frame, the length of the rocket gets contracted, since it is moving, and given by

$$l = l_0\sqrt{1-(v/c)^2} \text{ and } l'' = l_0\sqrt{1-(v''_r/c)^2} = l_0\frac{1-(v/c)^2}{1+(v/c)^2},$$

respectively. In the S frame, the time needed for the meteorite to cross the rocket is given by

$$\Delta T = \frac{l}{2v} = \frac{l_0\sqrt{1-(v/c)^2}}{2v}$$

in S' frame

$$\Delta T' = \frac{l_0}{v'_m} = \frac{l_0\left[1+(v/c)^2\right]}{2v}$$

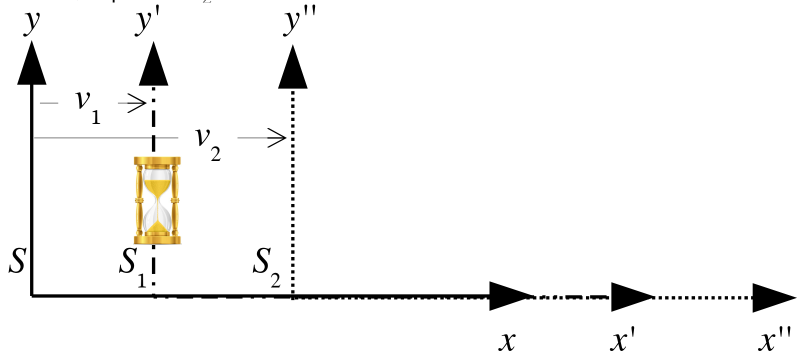
and in S'' frame

$$\Delta T'' = \frac{l''}{v''_r} = \frac{l_0\left[1-(v/c)^2\right]}{2v}$$

It is worth nothing that $\Delta T''$ corresponds to the proper time. Two events we are considering can be described as, the meteorite meets the beginning of the rocket and the meteorite meets the end of the rocket, respectively. In S'', those two events happen at the same spatial position, i.e. at the coordinate of the meteorite which is at rest.

Problem 3

Two inertial frames S_1 and S_2 are moving with velocity v_1 and v_2 along the x-axis in the positive direction, with respect to a third inertial frame, S. A sand clock is placed in the S_1 frame at $x_1 = 0$ and kept at rest in the S_1 frame. The observer in the S frame saw that at time t all the sand dropped to the bottom. Calculate the time when the observer in the S_2 frame saw that all the sand dropped to the bottom, as a function of t, v_1 and v_2.



Solution

The time when all the sand dropped seen by an observer in the S_2 frame, t_2, is given by the Lorentz transformation:

$$t_2 = \frac{t - v_2 x/c^2}{\sqrt{1-(v_2/c)^2}}$$

where x is the position of the sand clock seen by the observer in the S frame. The Lorentz transformation for x is given by

$$x = \frac{x_1 + v_1 t_1}{\sqrt{1-(v_1/c)^2}} = \frac{v_1 t_1}{\sqrt{1-(v_1/c)^2}}$$

where t_1 is the time when all the sand dropped seen by an observer in the S_1 frame. Two times t_1 and t are related through the Lorentz transformation as

$$t_1 = \frac{t - v_1 x/c^2}{\sqrt{1-(v_1/c)^2}}$$

By inserting this in the expression x, it follows that

$$x = \frac{v_1 t_1}{\sqrt{1-(v_1/c)^2}} = \frac{v_1\left(t - v_1 x/c^2\right)}{1-(v_1/c)^2} = \frac{v_1 t - x(v_1/c)^2}{1-(v_1/c)^2}$$

which could be rewritten as

$$x\left(1-(v_1/c)^2\right) = v_1 t - x(v_1/c)^2$$

Therefore, $x = v_1 t$ and from the expression of t_2, we obtain

$$t_2 = \frac{t - v_2 x/c^2}{\sqrt{1-(v_2/c)^2}} = \frac{t - v_1 v_2 t/c^2}{\sqrt{1-(v_2/c)^2}} = \frac{1 - v_1 v_2/c^2}{\sqrt{1-(v_2/c)^2}}\, t.$$