General Physics II at EPFL

(2017-2018 SS, Wed 17:15-19:00 and Thu 9:15-10:00, Exercise Thu 10:15-12:00)

Special Relativity (4th week)

Lorentz Invariance and Causality

As shown before, the Lorentz transformation of the space-time in the x-direction:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}.$$

It follows that

$$\begin{aligned} c^2t'^2 - x'^2 - y'^2 - z'^2 &= \gamma^2\left(ct - \beta x\right)^2 - \gamma^2\left(x - \beta ct\right)^2 - y^2 - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2 \end{aligned}$$

Therefore, four dimensional interval between the two events, $E_1(t_1, x_1, y_1, z_1)$ and $E_2(t_2, x_2, y_2, z_2)$, defined as

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

where $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, and $\Delta z = z_2 - z_1$, remains invariant under Lorentz transformation.

Note that for $\Delta x = \Delta y = \Delta z = 0$, $\Delta t \equiv \Delta t_0$ is the time interval for events occurring at the same space point, i.e. proper time and

$$\Delta s^2 = c^2\Delta t_0^2 \text{, i.e. } \Delta t_0 = \frac{\sqrt{\Delta s^2}}{c}$$

Thus, proper time is identical to the space-time distance between the two events (divided by c), if $\Delta s^2 \geq 0$. For two events with $\Delta s^2 < 0$, proper time becomes imaginary thus cannot be defined. Note that proper time of a particle corresponds to the time measured with a clock attached the particle.

Length, l, can considered as the spatial distance between the two space points that are relatively at rest, measured simultaneously, i.e. $\Delta t = 0$:

$$\Delta s^2 = -\Delta x^2 - \Delta y^2 - \Delta z^2 = -l^2 \text{, i.e. } l = \sqrt{-\Delta s^2}$$

Thus l, identical proper length, can be meaningful only when $\Delta s^2 \leq 0$.

Let us imagine a two dimensional space-time, ct and x. Such a space can be illustrated by a two dimensional Cartesian coordinate system (ct, ix). From a given point, $E_1(t_1, x_1)$, light propagates as a straight line,

$$t - t_1 = \pm\frac{1}{c}\left(x - x_1\right)$$

Any points $E_i(t_i, x_i)$ that is

$$\left|t_2 - t_1\right| \geq \frac{1}{c}\left|x_2 - x_1\right|$$

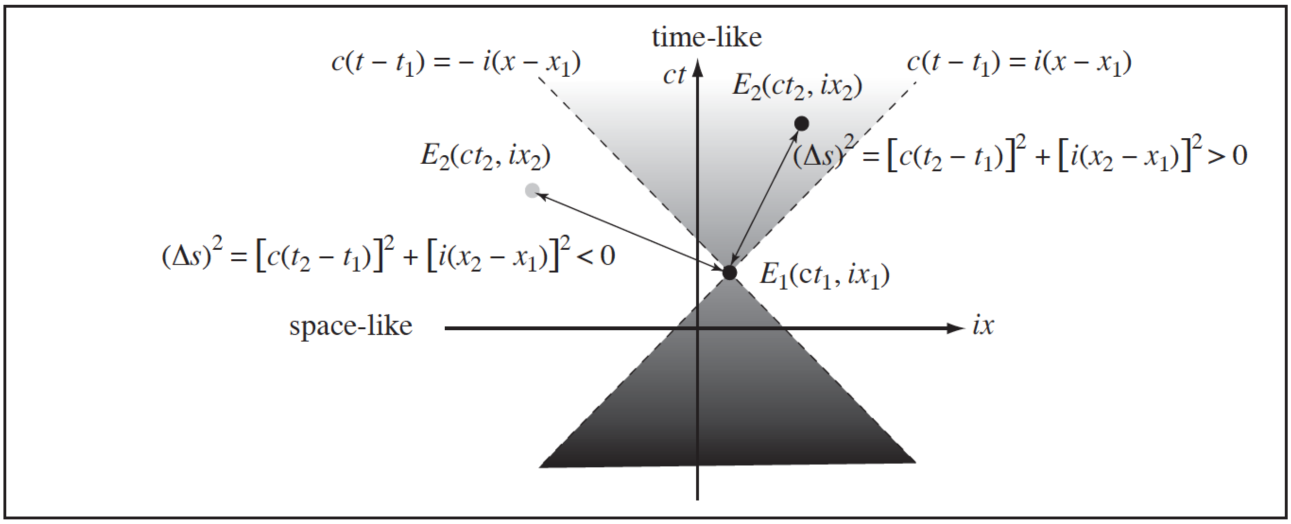
in the area shown in Figure 7, can be reached from E_1 by moving with a velocity

$$v = \frac{x_2 - x_1}{t_2 - t_1} \leq c\,.$$

Those points are referred as time-like. Proper time is then for events which are time like.

In order to reach any point out side of the area, one needs to move faster than light, and referred as space-like. From the fact $\gamma = 1/\sqrt{1-(v/c)^2}$ becomes imaginary for $v > c$, one can already speculate that no one can move faster than the speed of light. In this case, two space-like events are causally unrelated, i.e. one cannot influence the other, since any interaction between the two points cannot propagate faster than light. So only the time-like events can be causally related.

Figure 7



Energy Momentum Four Vector and Lorentz Transformation

We introduce a quantity τ as

$$\tau = \frac{1}{c}\sqrt{c^2t^2 - x^2 - y^2 - z^2}$$

which then invariant under the Lorentz transformation as discussed in the previous section and has a dimension of time. It follows that

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

thus,

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2} = \sqrt{1-\left(\frac{u}{c}\right)^2}\ .$$

Let us consider a quantity, $d\vec{x}/d\tau$. It follows that

$$\frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \frac{\vec{u}}{\sqrt{1-(u/c)^2}}$$

Therefore, the relativistic momentum derived before

$$\vec{p} = \frac{m_0\vec{u}}{\sqrt{1-(|\vec{u}|/c)^2}}$$

can be written as

$$\vec{p} = m_0 \frac{d\vec{x}}{d\tau}$$

Similarly for the energy, we can derive

$$\frac{E}{c} = m_0 \frac{d(ct)}{d\tau} = \frac{m_0 c}{\sqrt{1-(u/c)^2}} .$$

Since $(ct,\ x,\ y,\ z)$ is a Lorentz vector, τ a Lorentz invariant quantity and m_0 a constant, a four vector defined as $(E/c,\ p_x,\ p_y,\ p_z)$ is transformed as a Lorentz vector. Therefore, the energy-momentum four-vector is called "Lorentz four-vector". Since it is a Lorentz four-vector, the four dimensional interval, $E^2/c^2 - p^2$, is invariant under the Lorentz transformation; i.e. E and p will have different values in different inertial frames, $E^2/c^2 - p^2$ remains invariant. From the energy-momentum relation $E^2 = m_0^2 c^4 + p^2 c^2$, it follows that

$$E^2/c^2 - p^2 = m_0^2 c^2$$

thus the rest mass, m_0, is a Lorentz invariant quantity, called a Lorentz scalar. The mass given by

$$m_0 = \frac{\sqrt{E^2/c^2 - p^2}}{c}$$

is often called "invariant mass", since it is invariant under the Lorentz transformation.

Another way to obtain the result

This can also be shown by explicitly transforming the energy and momentum of a particle with a velocity $\vec{u}$ in a frame S to those given in another frame S', which is moving with a constant velocity v respect to S along the positive x direction. The energy and momentum in S is given by

$$\frac{E}{c} = \frac{m_0 c}{\sqrt{1-(|\vec{u}|/c)^2}},\ \vec{p} = \frac{m_0 \vec{u}}{\sqrt{1-(|\vec{u}|/c)^2}}$$

From the velocity transformation, the velocity of the particle in the frame S, $\vec{u}$, is given by the velocity in S', $\vec{u}'$, as

$$u_x = \frac{u'_x + c\beta}{1+\beta u'_x/c},\ u_y = \frac{1}{\gamma}\frac{u'_y}{1+\beta u'_x/c},\ u_z = \frac{1}{\gamma}\frac{u'_z}{1+\beta u'_x/c},\ \text{where } \beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

which allow to derive the energy and momentum given in S', by inserting those expressions to E and $\vec{p}$ given in terms of $\vec{u}$.

For this purpose, we need to calculate

$$\begin{aligned}
1 - \frac{1}{c^2}\left(u_x^2 + u_y^2 + u_z^2\right) &= 1 - \frac{1}{(1+\beta u'_x/c)^2}\left[\left(\frac{u'_x}{c} + \beta\right)^2 + \frac{u'^2_y + u'^2_z}{\gamma^2 c^2}\right] \\
&= \frac{(1+\beta u'_x/c)^2 - (u'_x/c + \beta)^2 - (1-\beta^2)\left(u'^2_y + u'^2_z\right)/c^2}{(1+\beta u'_x/c)^2} \\
&= \frac{(1-\beta^2) - (1-\beta^2)\left(u'^2_x + u'^2_y + u'^2_z\right)/c^2}{(1+\beta u'_x/c)^2} \\
&= \frac{1 - |\vec{u}'|^2/c^2}{\gamma^2(1+\beta u'_x/c)^2}
\end{aligned}$$

It follows that

$$\frac{1}{\sqrt{1-\left(|\vec{u}|/c\right)^2}} = \sqrt{\frac{\gamma^2\left(1+\beta u'_x/c\right)^2}{1-|\vec{u}'|^2/c^2}}$$

$$= \gamma \frac{1+\beta u'_x/c}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}}$$

Therefore,

$$\frac{E}{c} = \frac{m_0 c}{\sqrt{1-\left(|\vec{u}|/c\right)^2}} = \gamma \frac{m_0 c\left(1+\beta u'_x/c\right)}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}} = \frac{\gamma m_0 c + \beta\gamma m_0 u'_x}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}}$$

and

$$p_x = \frac{m_0 u_x}{\sqrt{1-\left(|\vec{u}|/c\right)^2}} = \gamma \frac{m_0\left(1+\beta u'_x/c\right)}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}} \frac{u'_x + c\beta}{1+\beta u'_x/c} = \frac{\gamma m_0 c\beta + \gamma m_0 u'_x}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}}$$

$$p_{y,z} = \frac{m_0 u_{y,z}}{\sqrt{1-\left(|\vec{u}|/c\right)^2}} = \gamma \frac{m_0\left(1+\beta u'_x/c\right)}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}} \frac{1}{\gamma} \frac{u'_{y,z}}{1+\beta u'_x/c} = \frac{\gamma m_0 u'_{y,z}}{\sqrt{1-\left(|\vec{u}'|/c\right)^2}}$$

i.e.

$$\frac{E}{c} = \gamma \frac{E'}{c} + \beta\gamma p'_x,\; p_x = \beta\gamma \frac{E'}{c} + \gamma p'_x, p_{y,z} = p'_{y,z}$$

or in a matrix notation

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix}$$

which is the Lorentz transformation.

Doppler Effect

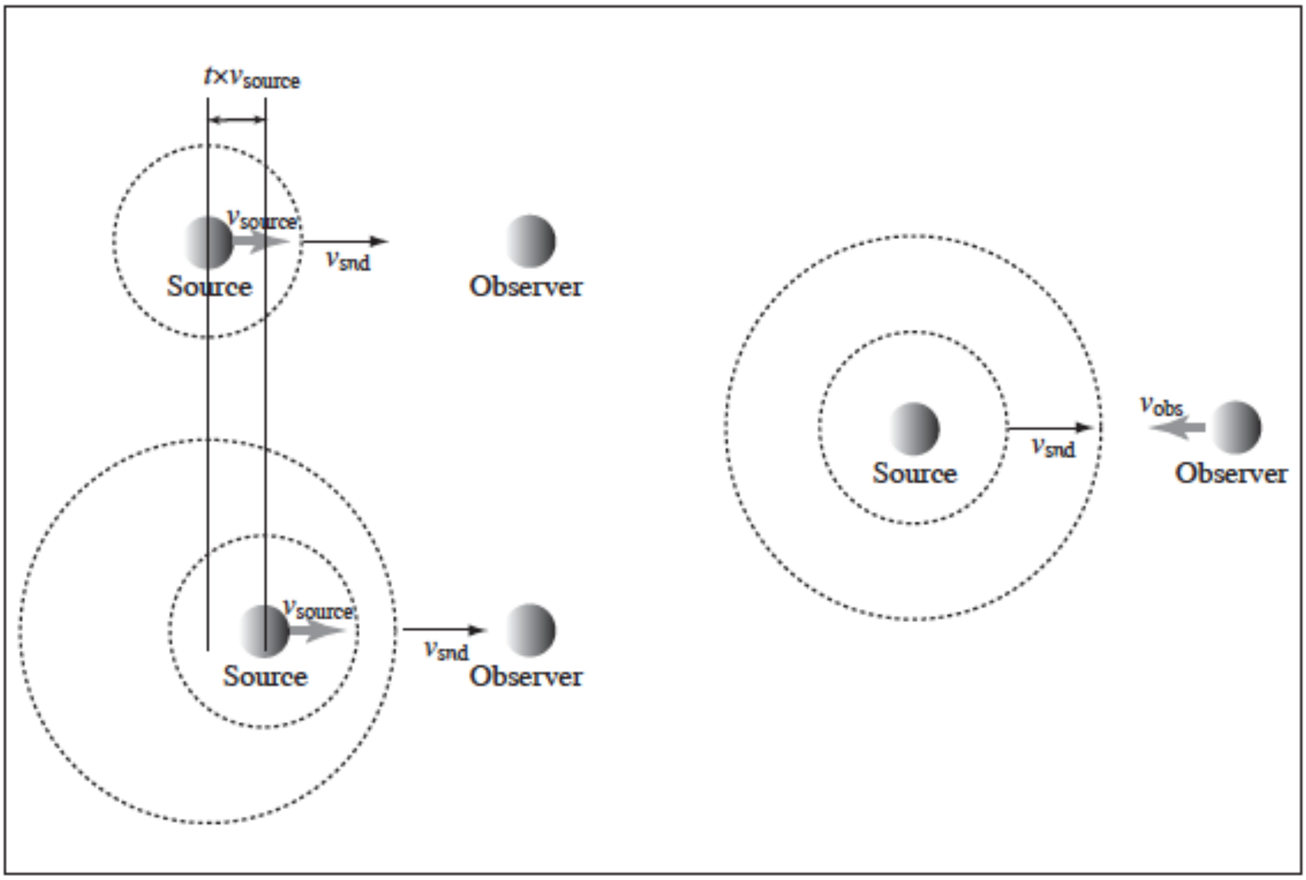
Doppler effect for sounds

Figure 10

Sounds are propagated through media, e.g. the air, and in the rest frame of the media, the speed is constant, v_{snd}. Therefore, it is most convenient to take this as our reference frame. The frequency, f, and wave length, λ, are related as

$$\lambda = \frac{v_{\text{snd}}}{f}$$

Figure 10



The source is approaching toward the observer with a velocity v_{source} and at $t = 0$, the crest of the sound left the source. The next crest leaves the source at $t = 1/f$ and the source has moved toward the observer by

$$t \times v_{\text{source}} = \frac{v_{\text{source}}}{f} = \lambda \frac{v_{\text{source}}}{v_{\text{snd}}}$$

Therefore, the distance between the two crests, which is the wave length seen by the observer, λ', is given by

$$\lambda' = \lambda - t \times v_{\text{source}} = \lambda\left(1 - \frac{v_{\text{source}}}{v_{\text{snd}}}\right)$$

i.e. the wavelength gets shorter, and the difference is proportional to the speed of the source. The frequency seen by the observer is then

$$f' = \frac{v_{\text{snd}}}{\lambda'} = v_{\text{snd}} \Big/ \lambda\left(1 - \frac{v_{\text{source}}}{v_{\text{snd}}}\right) = \frac{v_{\text{snd}}^2}{\lambda\left(v_{\text{snd}} - v_{\text{source}}\right)} = f\frac{v_{\text{snd}}}{\left(v_{\text{snd}} - v_{\text{source}}\right)}$$

i.e. the frequency the observer hears is higher than that emitted by the source. If the source is moving away from the observer, we need to replace v_{source} with $-v_{\text{source}}$.

Now, we consider a case where the observer is approaching toward the source with a velocity v_{obs}. The observer sees the sound approaching with a speed $v_{\text{sund}} + v_{\text{obs}}$, while the wavelength is unchanged. Therefore,

$$f' = \frac{v_{\text{snd}} + v_{\text{obs}}}{\lambda} = \frac{v_{\text{snd}}}{\lambda v_{\text{snd}}}\left(v_{\text{snd}} + v_{\text{obs}}\right) = \frac{f\left(v_{\text{snd}} + v_{\text{obs}}\right)}{v_{\text{snd}}}$$

i.e. the frequency the observer hears is higher than that emitted by the source. If the observer is moving away from the source, we need to replace v_{obs} with $-v_{\text{obs}}$.
By combining the two, we get a universal formula for the Doppler shift

$$f' = f\frac{v_{\text{snd}} \pm v_{\text{obs}}}{v_{\text{snd}} \mp v_{\text{source}}}$$

where the upper signs for the case where the movements decrease the distance between the source and observer and the lower signs increase. Note that both observer at rest and source moving, and observer at rest and source moving produce a similar effect but and effects are quantitatively not identical. Also note that we consider the cases where $v_{\text{snd}} > v_{\text{obs}}$ and $v_{\text{snd}} > v_{\text{source}}$.

<u>Doppler effect for light</u>
Unlike the case for the sound, the light propagate without media and the special relativity tells us that the velocity of the light, c, is identical for any observers moving with a constant speed respect to each other. We consider a case where a light source and a light sensor are approaching to each other with a constant velocity, v. We introduce an inertial frame, S', where the observer is at rest and the light source approaching with a constant velocity, v. In S' the time interval between the departures of the wave fronts is given by $\Delta t' = 1/f'$ where f', is the frequency of the light in S'. The first wave front leaves the source at $t = 0$. Then the second wave front leaves at $t = \Delta t'$. When the source emits the second wave front, it moved toward the observer by $v\Delta t'$. Therefore, the distance between the two wave fronts in S' is given by

$$(c - v)\Delta t'$$

which corresponds to the wavelength of the light measured in S', λ'.

In another inertial frame, S, where the light source is at rest, the emissions of the wave fronts take place at the same space coordinate. Therefore the time intervals between the emissions, $\Delta t = 1/f$ is proper time, where f is the frequency of the light measured in S. Proper time receives time dilation and the time interval in S' is seen as

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - (v/c)^2}}$$

Note that the S' is moving with a constant velocity respect to S.

It follows that

$$\lambda' = (c - v)\Delta t' = \frac{c - v}{\sqrt{1 - (v/c)^2}}\Delta t = \sqrt{\frac{(c - v)^2}{c^2 - v^2}}c\Delta t = \sqrt{\frac{c - v}{c + v}}\lambda$$

where λ is the weave length measured in S. For the frequency, we have

$$f' = \frac{c}{\lambda'} = \sqrt{\frac{c + v}{c - v}}f$$

In conclusion, the source and observer are **approaching** with a relative velocity, v, the frequency and wavelength of the light seen by observer are given by

$$f' = \sqrt{\frac{c + v}{c - v}}f \text{ and } \lambda' = \sqrt{\frac{c - v}{c + v}}\lambda'$$

i.e. the frequency is shifted to higher (blue shift). If two are **moving apart**

$$f_O = \sqrt{\frac{c - v}{c + v}}f_S \text{ and } \lambda_O = \sqrt{\frac{c + v}{c - v}}\lambda_S$$

frequency is sifted lower (red shift).

Astronomer found that the frequencies of lights from the distant galaxies are shifted toward lower frequencies. Thus the distance between the Earth and galaxies are increasing, i.e. our universe is expanding.

Transverse Doppler effect

For the sound, the Doppler effect is purely due to the velocity subtraction and addition between the sound velocity and the velocities of the source and observer. If the direction of the sound emitted is perpendicular to the direction of the movement of the source or observer, there is no Doppler effect. In the relativity, time dilation that does not depend on the direction of the movement, also plays a role. Therefore, there is still a Doppler effect even if the light is emitted perpendicular to the direction of the source (or the observer) movement.