General Physics II at EPFL

2018-2019 SS: Lecture; Wed 17:15-19:00 and Thu 8:15-9:00, Special session; 9:15-10:00

Exercise; Thu 10:15-12:00

Special Relativity (2nd week)

Time Dilatation and Length Contraction

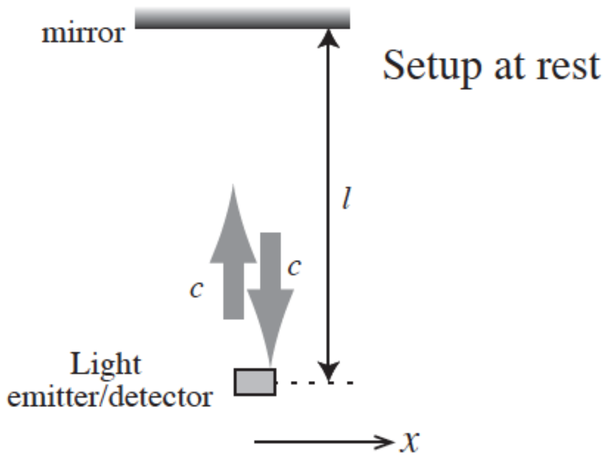
Time Dilatation

Proper time is defined as the time interval between two events took place at the same space coordinate. The same events seen in a different frame S' do not happen at the same space coordinate. For example, consider a case where a particle is at rest in the S frame. We call the time interval between the creation of the particle and its decay as "decay time". In the S frame, the decay time is a *proper time*, since the two events happened at the same space coordinate. Now we consider S' frame which is moving with a constant speeded v in x direction. In the S' frame, the particle moves backward in x direction with a velocity v. Therefore, the two events do not happen at the same space coordinates in the S' frame.

It turns out that the decay time measured in S' frame, i.e. the decay particle of a moving particle, is longer than that measured in the S frame where particle was at rest, i.e. the time associated to the moving frame evolves slower when compared to the time of the frame where the clock is at rest.

Gedanken Experiment

We consider Event-1 to be the emission of the light. The light is reflected with a mirror and seen by a detector, which is called Event-2. In the S frame, the emitter and detector are at the same coordinate, as shown in the figure below.

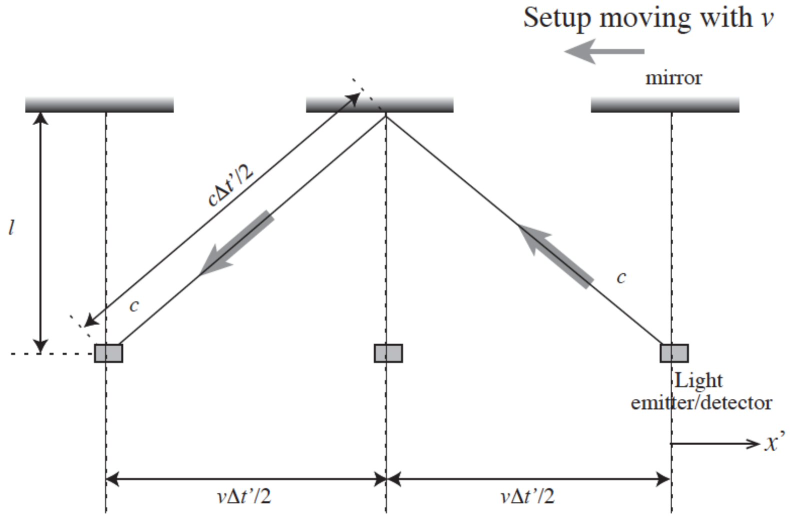


Experimental set up seen in S frame.

Since the two events, Event-1 and Event-2, take place at the same coordinate, the time interval between the two is a proper time and given by

$$\Delta t_0 = t_2 - t_1 = \frac{2l}{c}$$

In S', frame which is moving with a constant velocity, v, in x direction with respect to S frame, the light emitter, mirror and detector are moving with a constant velocity, v, in the backward direction of x, as seen in the figure below.



Experimental setup seen in S' frame.

In S' frame, the time interval between the two events $\Delta t'$ can be derived from

$$\left(c\times\frac{\Delta t'}{2}\right)^2 = l^2 + \left(v\times\frac{\Delta t'}{2}\right)^2 \quad\Rightarrow\quad \Delta t'^2 = 4\frac{l^2}{c^2} + \left(\frac{v}{c}\right)^2 \Delta t'^2$$

$$\Delta t' = \frac{2l/c}{\sqrt{1-(v/c)^2}} = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \gamma \Delta t_0$$

I.e. time interval between the two events measured in the frame where the setup is at rest, Δt_0, is smaller than that measured in the frame where the set-up is moving.

This effect can be shown more formally using the Lorentz transformation. This will be for your exercise.

Directionality: Time dilatation happens when the object moves in any direction.

<u>Length Contraction</u>

Length: spatial distance between the two space points, which are stationary relatively, or distance between the two spatial points measured at the same time.

Proper length: length between the two space points measured in the frame where the two points are at rest.

Assume that in the S frame the earth and moon are at rest. In the S frame, a space ship travels with a velocity, v. We then have the following definitions:

- Event $E_1(\vec{r}_E, t_1')$ the space ship leaves the earth where $\vec{r}_E$ and t_1' are the coordinate of the earth and the departure time respectively in S frame.
- Event $E_2(\vec{r}_M, t_2')$ the space ship arrives at the moon, where $\vec{r}_M$ and t_2' are the coordinate of the moon and the arrival time respectively, measured in the S frame.

Then the spatial distance and time interval of the two events in the S frame are given by

$$l_0 \equiv \left|\vec{r}_\mathrm{M} - \vec{r}_\mathrm{E}\right|,\ \Delta t = t_2' - t_1' = \frac{l_0}{v}$$

which is the proper length from the definition.

We then consider S' frame, which is moving with a constant velocity with respect to S frame so that the space ship is at rest in that frame: i.e. the earth and moon are moving with a velocity, v, in opposite direction respect to the direction of the S' movement respect to S. In S' frame, the space ship departure is given by $E_1\left(\vec{r}_\mathrm{w},\ t_1\right)$ where $\vec{r}_\mathrm{W}$ and t_1 are the coordinate of the space ship and the time when the position of the moving earth overlaps with that of the space ship, respectively. Similarly the arrival at the moon is given by $E_2\left(\vec{r}_\mathrm{w},\ t_2\right)$ is in S' frame where t_2 is the time when the position of the moving moon overlaps with that of the space ship,. Since in the S' the space ship is at rest. Hence the tine interval between the two events,

$$\Delta t_0 = t_2 - t_1$$

is a proper time. The time interval of the two events S is then given by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}}$$

Using the time interval between the two events and their common velocity, the distance between the earth and the moon in S' is given by

$$l = v\Delta t_0 = v\Delta t\sqrt{1-(v/c)^2} = l_0\sqrt{1-(v/c)^2} = \frac{l_0}{\gamma}$$

The distance between the earth and moon measured in S' frame where the earth and moon are moving with a common velocity v is shorter by $\sqrt{1-(v/c)^2}$ than the that measured in S frame where the earth and moon are at rest: i.e. the length of a moving object is contracting compared to the proper length.

The length can be shown in a more formal way using Lorentz transformation. This is demonstrated in the special session.

Directionality: Contraction occurs only along the direction of motion. The length perpendicular to the direction of motion is unchanged.

Values of γ

v	0	$0.01c$	$0.10c$	$0.50c$	$0.90c$	$0.99c$
γ	1.000	1.000	1.005	1.15	2.3	7.1

NB: $0.01c$ = 3000 km/sec

Relativistic Momentum

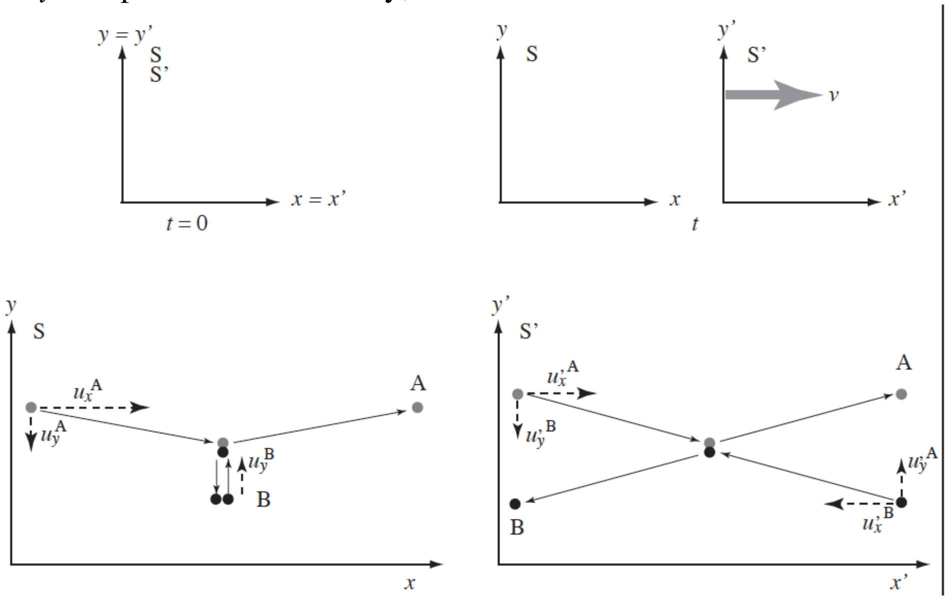
Momentum is given in Newtonian mechanics by

$$p = mu$$

where m is the mass of the object and u velocity. Let us consider an elastic collision of the two objects, A and B, in two-dimensional space, x-y. In the S' fame, they are moving with a velocity $\vec{u}'^{A} = \left(u'^{A}_x, u'^{A}_y\right)$ and $\vec{u}'^{B} = \left(u'^{B}_x, u'^{B}_y\right)$ and making an elastic collision symmetrically i.e. the y-component of the velocity for A becomes $-u'^{A}_y$ and B, $-u'^{B}_y$, thus after the collision, their velocity become $\left(u'^{A}_x, -u'^{A}_y\right)$ and $\left(u'^{B}_x, -u'^{B}_y\right)$. If the two objects have an identical mass, m, the y-component of the total momentum before the collision is $mu'^{A}_y + mu'^{B}_y$ and after the collision is $-mu'^{A}_y - mu'^{B}_y$. Momentum conservation leads to $mu'^{A}_y + mu'^{B}_y = -mu'^{A}_y - mu'^{B}_y$ i.e.

$$u'^{A}_y = -u'^{B}_y$$

Now introduce another inertial frames, S, where at $t = 0$, the two systems overlap each other, and S' moves with a constant velocity, v, in a positive x direction. Galilean transformation of velocity gives no change in the y-component of the velocity, hence the momentum conservation holds in S.



The velocity v is chosen such that the object B is moving only in y-direction in S i.e. $\vec{u}^{A} = \left(u^{A}_x, u^{A}_y\right)$ and $\vec{u}^{B} = \left(0, u^{B}_y\right)$. As seen before, the y-component (non boosted) of the velocity is also affected in the special relativity. Lorentz transformation for velocity gives the velocities for A and B before the collisions to be

$$\vec{u}'^{A} = \left(\frac{u^{A}_x - v}{1 - vu^{A}_x/c^2}, \frac{u^{A}_y\sqrt{1-\beta^2}}{1 - vu^{A}_x/c^2}\right), \quad \vec{u}'^{B} = \left(-v, u^{B}_y\sqrt{1-\beta^2}\right)$$

where $\beta = v/c$. Since $u'^{A}_y = -u'^{B}_y$,

$$\frac{u^{A}_y\sqrt{1-\beta^2}}{1 - vu^{A}_x/c^2} = -u^{B}_y\sqrt{1-\beta^2}$$

$$\therefore u^{B}_y = \frac{-u^{A}_y}{1 - vu^{A}_x/c^2}$$

i.e. $u_y^A \neq -u_y^B$. If momentum is defined as $p = mu$, $p_y^A \equiv mu_y^A \neq -mu_y^B \equiv -p_y^B$, and the momentum conservation law is not valid in S. Let us assume that A and B have different masses in S, such that momentum conservation is reinstalled: $p_y^A \equiv m_\mathrm{A} u_y^A = -m_\mathrm{B} u_y^B \equiv -p_y^B$.

$$m_\mathrm{A} u_y^\mathrm{A} = -m_\mathrm{B} u_y^\mathrm{B} = \frac{m_\mathrm{B} u_y^\mathrm{A}}{1 - v u_x^\mathrm{A} / c^2}$$

i.e.

$$m_\mathrm{A} = \frac{m_\mathrm{B}}{1 - v u_x^\mathrm{A} / c^2}$$

thus the mass appears to be different for different velocities. The mass of the object at rest, B, is often called as "rest mass" and we refer this as m_0.

Since the collision is symmetric in S', we have $u_x'^\mathrm{A} = -u_x'^\mathrm{B} = v$, i.e.

$$u_x'^\mathrm{A} = \frac{u_x^\mathrm{A} - v}{1 - v u_x^\mathrm{A} / c^2} = v$$

It follows that

$$\frac{u_x^\mathrm{A} - v}{1 - v u_x^\mathrm{A} / c^2} = v$$

$$u_x^\mathrm{A} - v = v - \frac{u_x^\mathrm{A}}{c^2} v^2$$

$$v^2 - 2 \frac{c^2}{u_x^\mathrm{A}} v + c^2 = 0$$

The velocity v is then given by

$$v = \frac{c^2}{u_x^\mathrm{A}} \pm \sqrt{\left(\frac{c^2}{u_x^\mathrm{A}} \right)^2 - c^2}$$

$$= \frac{c^2}{u_x^\mathrm{A}} \left[1 \pm \sqrt{1 - \left(u_x^\mathrm{A} / c \right)^2} \right]$$

Thus

$$1 - v \frac{u_x^\mathrm{A}}{c^2} = 1 - \left[1 \pm \sqrt{1 - \left(u_x^\mathrm{A} / c \right)^2} \right]$$

$$= \mp \sqrt{1 - \left(\frac{u_x^\mathrm{A}}{c} \right)^2}$$

and we chose the solution which gives a positive mass and

$$m_\mathrm{A} = \frac{m_\mathrm{B}}{\sqrt{1 - \left(u_x^\mathrm{A} / c \right)^2}}$$

For infinitesimally small u_y^A, we have $\left| u_x^A \right| = \left| \vec{u}^\mathrm{A} \right|$, and

$$m(|\vec{u}|) = \frac{m_0}{\sqrt{1 - \left(|\vec{u}| / c \right)^2}}$$

In conclusion, relativistic momentum is given by

$$\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}} \; .$$

Newton's second law is also given by

$$\vec{F} = \frac{d}{dt}\vec{p}$$

where $\vec{F}$ is the force. It follows that

$$\frac{d}{dt}\vec{p} = m_0 \frac{d}{dt}\frac{\vec{u}}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}$$

$$= m_0 \frac{1}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}\frac{d\vec{u}}{dt} + m_0 \vec{u}\frac{d}{dt}\frac{1}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}$$

$$= m_0 \frac{1}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}\frac{d\vec{u}}{dt} + \frac{m_0}{c^2}\frac{|\vec{u}|\left(d|\vec{u}|/dt\right)}{\left[1 - \left(|\vec{u}|/c\right)^2\right]^{\frac{3}{2}}}\vec{u}$$

By introducing the acceleration, $\vec{a} = d\vec{u}/dt$, it follows that

$$F = m_0 \frac{1}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}\frac{d\vec{u}}{dt} + \frac{m_0}{c^2}\frac{|\vec{u}|\left(d|\vec{u}|/dt\right)}{\left[1 - \left(|\vec{u}|/c\right)^2\right]^{\frac{3}{2}}}\vec{u}$$

$$= \frac{m_0 \vec{a}}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}} + \frac{m_0 (\vec{a}\cdot \vec{u})}{c^2\left[1 - \left(|\vec{u}|/c\right)^2\right]^{\frac{3}{2}}}\vec{u} \; .$$

$$= \frac{m_0}{\sqrt{1 - \left(|\vec{u}|/c\right)^2}}\left[\vec{a} + \frac{(\vec{a}\cdot \vec{u})}{c^2\left[1 - \left(|\vec{u}|/c\right)^2\right]}\vec{u}\right]$$

The Ultimate Speed

When the speed of the object, u, approaches to that of light, we have $\sqrt{1-(u/c)^2} \rightarrow 0$ and the mass of the object given by

$$m = \frac{m_0}{\sqrt{1-(u/c)^2}}$$

gets larger an larger, thus more and more difficult to accelerate further. In the limit of $u = c$, we have $m = \infty$ and one needs infinite energy to further accelerate the object. Therefore, nothing can travel faster then the speed of light.

Special Session

Length contraction and Lorentz Transformation

Place a bar at rest in the frame S along the x-axis. The two end points of the bar in S are given by $(x_1, 0, 0)$ and $(x_2, 0, 0)$. Since the bar is at rest, $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are independent of time, t. Therefore, for the proper length of the bar, $l_0 = x_2(t_2) - x_1(t_1)$, t_1 and t_2 can be different. The two end pints seen by the frame S' are no longer stationary and changing as functions of time, t', but the relative velocity between the two is zero, thus have a fixed distance. The Lorentz transformation gives

$$x_1' = -\gamma\beta c t_1 + \gamma x_1 \qquad \mathrm{c}t_1' = \gamma c t_1 - \gamma\beta x_1$$
$$x_2' = -\gamma\beta c t_2 + \gamma x_2 \qquad \mathrm{c}t_2' = \gamma c t_2 - \gamma\beta x_2$$

and the length of the bar is given by $l = x_2'(t_2') - x_1(t_1')$, where the both coordinates must be measured at the same time, $t_1' = t_2' \equiv t_0'$ since they are moving. It follows that

$$l = x_2' - x_1' = (\gamma x_2 - \gamma\beta c t_2) - (\gamma x_1 - \gamma\beta c t_1) = \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1)$$

Using $\mathrm{c}t_1 = \gamma c t_0' + \gamma\beta x_1'$ and $\mathrm{c}t_2 = \gamma c t_0' + \gamma\beta x_2'$, we have $\mathrm{c}(t_2 - t_1) = \gamma\beta(x_2' - x_1')$, thus $l = \gamma l_0 - \gamma^2\beta^2 l$. It follows that

$$l(1 + \gamma^2\beta^2) = \gamma l_0$$

and

$$1 + \gamma^2\beta^2 = 1 + \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} = \gamma^2$$

leads to

$$\gamma^2 l = \gamma l_0$$

In conclusion, the length, l, of a moving object by a velocity v scales as

$$l = \frac{l_0}{\gamma} = \sqrt{1 - \left(\frac{v}{c}\right)^2}\, l_0$$

where proper length, l_0 is the length of the object at rest, which is the length contraction.