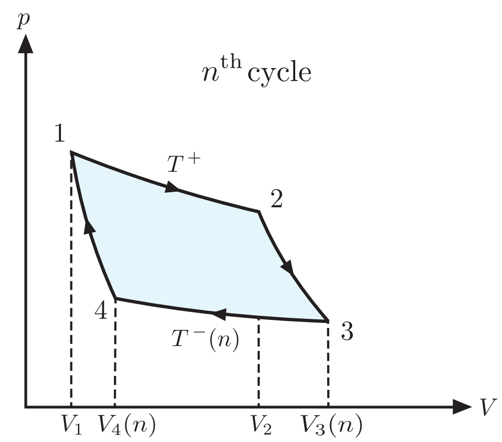
General instructions:

- You are given five sheets of A3 paper. The sheet with your name will be the envelop to put the other sheets inside of and return to us. Use the envelop sheet as scrap paper. However, cross all calculations on this envelop sheet out before you return the exam. Use each other sheet for a different problem in this exam.
- You must return these two pages of questions along with your exam inside the envelop.
- Circle your final answers to each question.
- Make sure you get all of the signs right. $+$: energy added to the system, -: energy taken out of the system.

Problem 1: Heat engine whose cold reservoir heats up in each round **40 points total**



Consider a Carnot cycle with two heat reservoirs. The engine consists of n moles of an ideal monoatomic gas. In addition, there is a hot heat reservoir always at temperature T^+. However, the cold 'reservoir' **heats up** in each cycle n due to the heat being dumped into it. (The reservoir is not strictly a reservoir since it is heating up, but we will continue to refer to it as a reservoir nevertheless.) We denote the temperature of the cold reservoir as $T^-(n)$. n refers to the number of cycles that have passed.

The steps of the cycle are as follows:

- $1 \to 2$ Isothermal expansion at temperature T^+ from V_1 to V_2
- $2 \to 3$ Adiabatic expansion from V_2 to $V_3(n)$
- $3 \to 4$ Isothermal compression at temperature $T^-(n)$ from $V_3(n)$ to $V_4(n)$
- $4 \to 1$ Adiabatic compression from $V_4(n)$ to V_1

For problems a)-g), we assume that the heat is dumped into the cold reservoir at the end of the cycle when the cycle arrives back at 1, not during the compression $3 \to 4$. In sub-problems h)-j), we study what actually happens if the reservoir heats up during the compression step $3 \to 4$, which is then no longer an isothermal process.

a) The cold and hot heat reservoirs have heat capacities denoted by C^- and C^+, respectively. Which of their values do you know and what are they? **2.5 points**

Solution:

$$C^+ = \infty$$

b) Draw the 4 paths corresponding to the 4 transformations of the cycle in a T-S diagram (temperature-vs.-entropy diagram). Indicate the directions of the arrows of the transformations. **2.5 points**

c) By comparing the adiabatic transformations in the cycle, show that the volumes satisfy the following identity: **5 points**

$$\frac{V_3(n)}{V_4(n)} = \frac{V_2}{V_1}$$

Solution: During the adiabatic expansion $2 \to 3$ and the abiabatic compression $4 \to 1$, the temperatures and volumes satisfy the following identities,

$$T^+ V_2^{\gamma-1} = T^-(n)\, V_3^{\gamma-1}(n) \qquad \text{et} \qquad T^+ V_1^{\gamma-1} = T^-(n)\, V_4^{\gamma-1}(n) \tag{1}$$

where $T_1 = T_2 = T^+$ et $T_3(n) = T_4(n) = T^-(n)$. The ratio of the identities (1) yields,

$$\left(\frac{V_2}{V_1}\right)^{\gamma-1} = \left(\frac{V_3(n)}{V_4(n)}\right)^{\gamma-1} \tag{2}$$

Thus, we obtain the identity,

$$\frac{V_2}{V_1} = \frac{V_3(n)}{V_4(n)} > 1 \tag{3}$$

d) What is the amount of heat $Q^-(n)$ transferred to the system in the transformation 3 $\to$ 4, and thus the heat $|Q^-(n)|$ dumped into the cold reservoir at the end of every cycle? Use the identity from sub-problem c) to get rid of $V_3(n)$ and $V_4(n)$ in your answer in favor of V_1 and V_2. **5 points**

Solution:

$$Q^-(n) = -NRT^-(n)\ln\left(\frac{V_2}{V_1}\right)$$

Solution 1 : During the isothermal compression of the n^{th} cycle, the internal energy variation of the ideal gas vanishes,

$$\Delta U_{34}(n) = 0 \tag{4}$$

Thus, according to the first law and taking into account the ideal gas equation of state,

$$p(n) = \frac{NRT^-(n)}{V(n)} \tag{5}$$

the heat $Q^-(n)$ restituted by the monoatomic ideal gas to the cold source during the isothermal compression at temperature $T^-(n)$ is the opposite of the work performed on the ideal gas,

$$Q^-(n) \equiv Q_{34}(n) = -W_{34}(n) = \int_{V_3(n)}^{V_4(n)} p(n)\, dV = NRT^-(n) \int_{V_3(n)}^{V_4(n)} \frac{dV}{V} \tag{6}$$

In view of the volume ratio (3), the heat $Q^-(n)$ becomes,

$$Q^-(n) = -NRT^-(n) \int_{V_1}^{V_2} \frac{dV(n)}{V(n)} = -NRT^-(n)\ln\left(\frac{V_2}{V_1}\right) < 0 \tag{7}$$

Solution 2 : During the n^{th} cycle, the heat $Q^-(n)$ is restituted by the monoatomic ideal gas to the cold source during the isothermal compression at temperature $T^-(n)$,

$$Q^-(n) \equiv Q_{34}(n) = \int_{S_3(n)}^{S_4(n)} T(n)\, dS(n) = T^-(n) \int_{S^+}^{S^-} dS \tag{8}$$

where $S_3(n) = S^+$ and $S_4(n) = S^-$. During this process, the Gibbs relation is written as,

$$dU(n) = T^-(n)\, dS - p(n)\, dV(n) = 0 \tag{9}$$

Thus, for the ideal monoatomic gas during the isothermal compression,

$$dS = \frac{p(n)\, dV(n)}{T^-(n)} \tag{10}$$

taking into account the equation of state,

$$p(n)\, V(n) = NRT^-(n) \tag{11}$$

the heat $Q^-(n)$ restituted to the cold source (8) becomes,

$$Q^-(n) = NRT^-(n) \int_{V_3(n)}^{V_4(n)} \frac{dV(n)}{V(n)} = -NRT^-(n)\ln\left(\frac{V_3(n)}{V_4(n)}\right) = -NRT^-(n)\ln\left(\frac{V_2}{V_1}\right) < 0 \tag{12}$$

e) Compute the work $W(n)$ performed on the system during the n^{th} cycle. **5 points**

Solution:

Solution 1 : Taking into account the result (12), the heat provided to the system during the isothermal expansion $1 \to 2$ is written as,

$$Q^{+} \equiv Q_{12} = NRT^{+} \int_{V_1}^{V_2} \frac{dV}{V} = NRT^{+} \ln\left(\frac{V_2}{V_1}\right) > 0 \tag{13}$$

Taking into account the fact that the processes $2 \to 3$ and $4 \to 1$ are adiabatic, there is no heat transfer during these processes,

$$Q_{23} = 0 \qquad \text{and} \qquad Q_{41} = 0 \tag{14}$$

Thus, taking into account the heats (12), (13) and (14), the heat provided to the system during the n^{th} cycle is,

$$Q(n) = Q_{12} + Q_{23} + Q_{34}(n) + Q_{41} = Q^{+} + Q^{-}(n) = NR\left(T^{+} - T^{-}(n)\right) \ln\left(\frac{V_2}{V_1}\right) > 0 \tag{15}$$

According to the first law applied to the n^{th} cycle, the worked performed on the environment during the n^{th} cycle is written as,

$$W(n) = \Delta U(n) - Q(n) = -Q(n) = -NR\left(T^{+} - T^{-}(n)\right) \ln\left(\frac{V_2}{V_1}\right) < 0 \tag{16}$$

Solution 2 : Applying the first law, taking into account the heat (13), the work performed on the environment during the isothermal expansion $1 \to 2$ of the n^{th} cycle is written as,

$$W^{+} \equiv W_{12} = \Delta U_{12} - Q_{12} = -Q^{+} = -NRT^{+} \ln\left(\frac{V_2}{V_1}\right) < 0 \tag{17}$$

the work performed on the environment during the isothermal expansion compression $3 \to 4$ of the n^{th} cycle reads,

$$W^{-}(n) \equiv W_{34}(n) = \Delta U_{34}(n) - Q_{34}(n) = -Q^{-}(n) = NRT^{-}(n) \ln\left(\frac{V_2}{V_1}\right) > 0 \tag{18}$$

Applying the first law, taking into account the heat (14), the work performed on the environment during the adiabatic expansion $2 \to 3$ of the n^{th} cycle is written as,

$$W_{23}(n) = \Delta U_{23}(n) - Q_{23}(n) = \Delta U_{23}(n) = C_V\left(T_3(n) - T_2\right) = -\frac{3}{2} NR\left(T^{+} - T^{-}(n)\right) < 0 \tag{19}$$

Applying the first law, taking into account the heat (14), the work performed on the system during the adiabatic compression $4 \to 1$ of the n^{th} cycle reads,

$$W_{41}(n) = \Delta U_{41}(n) - Q_{41}(n) = \Delta U_{41}(n) = C_V\left(T_1 - T_4(n)\right) = \frac{3}{2} NR\left(T^{+} - T^{-}(n)\right) > 0 \tag{20}$$

Thus, taking into account the works (17), (18), (19) and (20), the work performed on the environment during the n^{th} cycle is,

$$W(n) = W_{12} + W_{23}(n) + W_{34}(n) + W_{41}(n) = -NR\left(T^{+} - T^{-}(n)\right) \ln\left(\frac{V_2}{V_1}\right) < 0 \tag{21}$$

f) Show that the change in temperature $\Delta T^{-}(n)$ of the cold source during the n^{th} cycle can be expressed as

$$\Delta T^{-}(n) = \lambda T^{-}(n) \quad ,$$

and determine the coefficient λ. **5 points**

Solution:

$$\Delta T^{-}(n) = -\frac{Q^{-}(n)}{C^{-}} = \frac{NR \ln(V_2/V_1)}{C^{-}} T^{-}(n)$$

$$\lambda = \frac{NR \ln(V_2/V_1)}{C^{-}}$$

g) What is the temperature $T^{-}(n)$ in terms of the temperature of the cold reservoir before the first cycle $T^{-}(0)$? **5 points**

Solution:

$$T^{-}(n) = (1+\lambda)^{n}\, T^{-}(0) \tag{22}$$
$$T^{-}(n) = \left(1 + \frac{NR\ln\left(V_2/V_1\right)}{C^{-}}\right)^{n} T^{-}(0) \tag{23}$$

Now, let us stop thinking about the heat dumped into the cold reservoir at the end of each cycle. Instead, let us analyze what actually happens if the gas is in contact with the cold reservoir, which has heat capacity C^{-} and temperature T^{-} before the compression of the gas. Together, the gas and the reservoir are thermally insulated. As the piston is pushed in, work is done on the gas, which heats up and passes heat to the reservoir.

h) Relate an infinitesimal amount of work done on the gas to the infinitesimal change in temperature of the whole system. **2.5 points**

Solution:

$$-pdV = (C_V + C)dT \quad C = \frac{3}{2}k_B \tag{24}$$

i) Relate the infinitesimal change in the temperature of an ideal gas to infinitesimal changes in volume and pressure of the gas. In this way, relate the work done on the gas that you used in sub-problem h) to infinitesimal changes in pressure and volume. You should only have as infinitesimal quantities dp and dV in your equation. **2.5 points**

Solution:

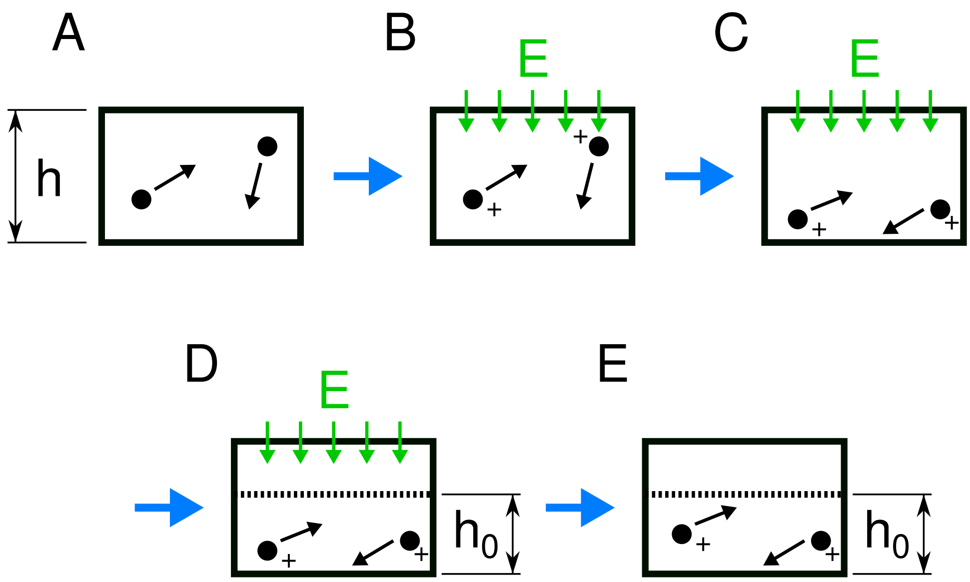
$$dT = \frac{1}{nR}\left(pdV + Vdp\right) \tag{25}$$
$$-pdV = (C_V + C)dT = \frac{C_{\text{tot}}}{nR}\left(pdV + Vdp\right) \tag{26}$$

j) Integrate this equation to see how p and V change during this transformation actually, as opposed to the idealization we used in sub-problems a) to g). You should end up with an equation relating p and V during this transformation. **5 points**

Solution:

$$-\left(\frac{nR}{C_{\text{tot}}} + 1\right) pdV = Vdp \tag{27}$$
$$-\left(\frac{nR}{C_{\text{tot}}} + 1\right) \frac{dV}{V} = \frac{dp}{p} \tag{28}$$
$$-\left(\frac{nR}{C_{\text{tot}}} + 1\right) \log\frac{V}{V_0} = \log\frac{p}{p_0} \tag{29}$$
$$\text{const.} = pV^{\frac{nR}{C_{\text{tot}}}+1} \tag{30}$$
$$\text{const.} = pV^{\frac{nR+C_{\text{tot}}}{C_{\text{tot}}}} \tag{31}$$

Problem 2: Statistics of particles in a box subject to an electric field **40 points total**



In this experiment, a monatomic gas of N atoms of mass m and charge q is in a thermally isolated box of volume V and height h. The following steps take place successively:

A. The gas is inside the box at temperature T_i.
B. An electric field of magnitude E is turned on, pointing down in the -z-direction. In this step, the field has just been turned on but nothing has changed inside the box yet.
C. The particles move in response to the electric field and reach equilibrium at temperature T_f.
D. A ceiling is put in at height h_0 from the bottom of the box.
E. The electric field is abruptly switched off.

Recall that the force $\vec{F}$ on a particle of charge q due to an electric field $\vec{E}$ is $\vec{F} = q\vec{E}$. The potential energy (U) of a particle with respect to the floor of the box is $U = qEz$, where $E = |\vec{E}|$ is the magnitude of the electric field. (Maybe it helps to remember a similar formula for the gravitational potential energy, which you have worked with before.)

a) In steps A or B, what is the probability density $p_i(z)$ of finding an atom at height z above the floor of the box? **5 points**

Solution:

$$p_i(z) = \frac{1}{h} \tag{32}$$

b) In steps A or B, what is the average position in z above the floor to find a particle? **5 points**

Solution:

$$\frac{h}{2} \tag{33}$$

c) In step B, what is the total kinetic and potential energy of the gas? **5 points**

Solution:

$$N\left[\frac{3}{2}k_BT_i + \frac{h}{2}qE\right] \tag{34}$$

d) In step C, what is the probability density $p_f(z)$ of finding an atom at height z above the floor? Suppose that the electric field is really strong so that the ceiling is effectively at $z = \infty$ after the gas equilibrates. (This means you can integrate to infinity, do not worry about the ceiling being there.) Also, for now, suppose you do not know the final temperature T_f. **5 points**

Solution:

$$p_f(z) = \frac{qE}{k_BT_f}\exp\left(-\frac{qEz}{k_BT_f}\right). \tag{35}$$

e) In step C, what is the total kinetic and potential energy of the gas? Assume the temperature is T_f, which you do not know yet. **5 points**

Solution:

$$\text{potential energy} = N\left[\frac{3}{2}k_BT_f + \int_0^\infty CqEze^{-Cz}dz\right] \quad \text{, where } C = \frac{qE}{k_BT_f} \tag{36}$$

$$= N\left[\frac{3}{2}k_BT_f + CqE\int_0^\infty ze^{-Cz}dz\right] \tag{37}$$

$$= N\left[\frac{3}{2}k_BT_f + CqE\frac{-d}{dC}\int_0^\infty e^{-Cz}dz\right] \tag{38}$$

$$= N\left[\frac{3}{2}k_BT_f + CqE\frac{1}{C^2}\right] \tag{39}$$

$$= N\left[\frac{3}{2}k_BT_f + \frac{qE}{C}\right] \tag{40}$$

$$= N\frac{5}{2}k_BT_f \tag{41}$$

f) Looking at the energies you calculated, what is T_f? **5 points**

Solution:

$$\frac{3}{2}k_BT_i + \frac{h}{2}qE = \frac{5}{2}k_BT_f \tag{42}$$

$$\frac{3}{5}T_i + \frac{h}{5k_B}qE = T_f \tag{43}$$

g) In step D, at what height h_0 must the ceiling be for only about one particle to be left above the ceiling? This means that you are asked to find h_0, so that the probability to be above h_0 is $1/N$. **5 points**

Solution:

$$1/N = \int_{h_0}^\infty p_f(z) = \exp\left(-\frac{qEh_0}{k_BT_f}\right) \tag{44}$$

$$h_0 = \frac{k_BT_f}{qE}\log N \tag{45}$$

h) Now, assume that you chose h_0 high enough so that all of your N particles are below the ceiling. In step E, you turn off the electric field. After you turn off the electric field and wait, what is the temperature T_e, the kinetic energy K_e, and potential energy U_e of the gas in the end? **5 points**

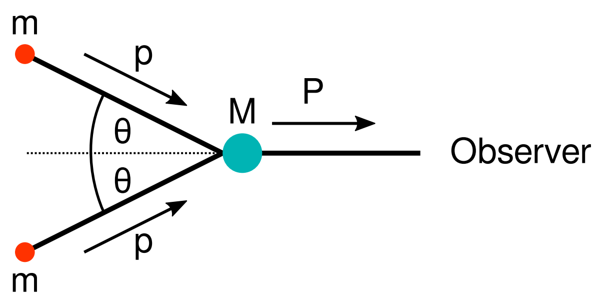
Solution: The situation is just like breaking a separating wall.

$$T_e = T_f \tag{46}$$

$$K_e = \frac{3}{2} k_B T_f \tag{47}$$

$$U_e = 0 \tag{48}$$

Problem 3: Relativistic collision at an angle **20 points total**



Two particles of known rest mass m are going with known total momentum p toward each other at a known angle θ as indicated in the diagram. They collide and form a particle of unknown rest mass M, which moves in the direction as indicated with unknown momentum P. For simplicity, assume that we work in units such that $c = 1$.

a) What is the momentum P and mass M of the new particle in terms of m, p, and θ? **5 points**

Solution:

$$P = 2p\cos\theta \tag{49}$$

$$E_{\text{initial}} = 2\sqrt{m^2 + p^2} \tag{50}$$

$$E_{\text{final}} = \sqrt{M^2 + P^2} = \sqrt{M^2 + 4p^2\cos^2\theta} \tag{51}$$

$$4m^2 + 4p^2 = M^2 + 4p^2\cos^2\theta \tag{52}$$

$$M^2 = 4m^2 + 4p^2\sin^2\theta \tag{53}$$

$$M = \sqrt{4m^2 + 4p^2\sin^2\theta} \tag{54}$$

b) What is the velocity U of the new particle in terms of m, p, and θ? **5 points**

Solution:

Solution 1:

Starting with:

$$E = M\gamma = \frac{M}{\sqrt{1-U^2}} \tag{55}$$

$$U = \sqrt{\frac{E^2 - M^2}{E^2}} = \frac{2p\cos\theta}{2\sqrt{m^2+p^2}} = \frac{p\cos\theta}{\sqrt{m^2+p^2}} \tag{56}$$

Solution 2:

Starting with:

$$P = \gamma M U \tag{57}$$

$$P^2 = \frac{M^2U^2}{1-U^2} \tag{58}$$

$$(M^2 + P^2)U^2 = P^2 \tag{59}$$

$$U = \frac{P}{E} = \frac{2p\cos\theta}{2\sqrt{m^2+p^2}} = \frac{p\cos\theta}{\sqrt{m^2+p^2}} \tag{60}$$

c) Suppose the new object of rest mass M hits a target. If you wanted as hard of an impact on the target as possible, what angle θ would you choose? **2.5 points**

Solution:

$$\theta = 0 \tag{61}$$

d) The new particle of mass M falls apart into two photons, one traveling in the same and one in the opposite direction of the direction of travel of the new particle. In the inertial frame of the new particle, the electromagnetic wave that the photons represent has frequency $f = M/2$. (If you know quantum mechanics, we set Planck's constant h equal to 1.) What is the frequency that someone standing in the path of the new particle will see in terms of the known parameters of the problem? Simplify your expression as much as you can. **5 points**

Solution:

$$f'^2 = \frac{1+U}{1-U}f^2 \tag{62}$$

$$= \frac{\sqrt{m^2+p^2} + p\cos\theta}{\sqrt{m^2+p^2} - p\cos\theta}\frac{M^2}{4} \tag{63}$$

$$= \frac{\sqrt{m^2+p^2} + p\cos\theta}{\sqrt{m^2+p^2} - p\cos\theta}\left(m^2 + p^2\sin^2\theta\right) \tag{64}$$

$$= \frac{(\sqrt{m^2+p^2} + p\cos\theta)^2}{m^2+p^2-p^2\cos^2\theta}\left(m^2 + p^2\sin^2\theta\right) \tag{65}$$

$$= \frac{(\sqrt{m^2+p^2} + p\cos\theta)^2}{m^2+p^2\sin^2\theta}\left(m^2 + p^2\sin^2\theta\right) \tag{66}$$

$$= (\sqrt{m^2+p^2} + p\cos\theta)^2 \tag{67}$$

$$f' = (\sqrt{m^2+p^2} + p\cos\theta) \tag{68}$$

e) At what angle θ does one have to shoot the two particles to obtain the highest frequency for the photon going toward the observer? **2.5 points**

Solution:

$$\theta = 0 \tag{69}$$