General Physics II: Tutorial Material

Lecture 8 (Chapter 8 Entropy)

1) Lets us consider a very large heat reservoir at a temperature T_R, and a small thermal system at T_S. The heat capacity of the small system is C (note that C is simply the amount of heat required to raise the temperature of an *entire* object or system by one degree, i.e. $Q = C\Delta T$). By putting them into thermal contact, they reach a thermal equilibrium at T_R, since the heat reservoir has such a large heat capacity and stays at the same temperature.

1. Calculate the entropy changes of the heat reservoir
2. Calculate the entropy changes of the small system.
3. Calculate the entropy changes of the total system.
4. Show that the change of the entropy of the total system is $\Delta S \geq 0$.

Solution: First we note that $Q = C\Delta T$. Thus, the heat of the small system and that of reservoir are given by $Q_{system} = C(T_R - T_S)$ and $Q_{reservoir} = -Q_{system}$, respectively. So the thermal energy flows from the reservoir to the small system when $T_R > T_S$. Process of the transferring of thermal energy from the heat reservoir itself can be considered as a reversible process since the process can be reversed by attaching another system with a temperature of $2T_R - T_S$. A similar argument can be valid for the system, since the process can be reversed by attaching the system to another heat reservoir with a temperature of T_S. However, the combined process of the heat reservoir and the system is irreversible.

1. Since the process of the heat reservoir alone is reversible and the temperature is fixed, we can use the formulas $Q = T\Delta S$ and $Q = C\Delta T$. The entropy change of the heat reservoir is then given by

   $$\Delta S_{reservoir} = \frac{Q_{reservoir}}{T_R} = C\frac{T_S - T_R}{T_R} \tag{1}$$

   Thus, change of the reservoir entropy is negative when $T_R > T_S$.

2. Since the process of the system alone is reversible and the temperature is not fixed, we can use the formulas $dQ = TdS$ and $dQ = CdT$. The entropy of the small system is then given by

   $$\Delta S_{system} = C\int_{T_S}^{T_R} \frac{dT}{T} = C\ln\frac{T_R}{T_S} \tag{2}$$

   Thus, the change of the system entropy is positive when $T_R > T_S$.

3. Entropy of the total system is given by

   $$\Delta S = \Delta S_{reservoir} + \Delta S_{system} = C\frac{(T_S - T_R)}{T_R} + C\ln\frac{T_R}{T_S} \tag{3}$$

4. Let us rewrite the change of the entropy as

   $$\Delta S = C\left(\frac{T_S}{T_R} - 1 - \ln\frac{T_S}{T_R}\right) \tag{4}$$

For $T_S/T_R = 0$ we obtain $\Delta S = \infty$, for $T_S/T_R = 1$ we get $\Delta S = 0$. From the derivative,

$$\frac{d\Delta S}{d(T_S/T_R)} = C\left(1 - \frac{T_R}{T_S}\right) \tag{5}$$

we learn that ΔS is monotonically decreasing function of T_S/T_R for $T_S/T_R = 0$ to $T_S/T_R = 1$. For $T_S/T_R = 1$ to $T_S/T_R = \infty$, ΔS is monotonically increasing as a function of T_S/T_R. Therefore, we can conclude that $\Delta S \geq 0$, and it is 0 when $T_R = T_S$.

2) Show that the entropy difference of an n-mol ideal gas, ΔS, when the state $A(P_1, V_1, T_1)$ is changed to $B(P_2, V_2, T_2)$ quasi-statically (i.e., reversible), is given by

$$\Delta S = nC_V \ln\frac{T_2}{T_1} + nR\ln\frac{V_2}{V_1} \tag{6}$$

Show that this leads to $\Delta S = 0$ for an adiabatic process, as expected from the definition.

Solution : From the first law of the thermodynamics, $Q = \Delta E_{int} + W$, where $W = P\Delta V$, and with the ideal gas law, $PV = nRT$ and $\Delta E_{int} = nC_V\Delta T$, heat can be written as

$$Q = \Delta E_{int} + P\Delta V = nC_V\Delta T + \frac{nRT}{V}\Delta V \tag{7}$$

For an infinitesimal quasi-static change of the state, this leads to

$$dQ = nC_V dT + \frac{nRT}{V}dV \tag{8}$$

and the change in the entropy becomes

$$dS = \frac{dQ}{T} = \frac{nC_V}{T}dT + \frac{nR}{V}dV \tag{9}$$

The entropy change for A to B is now given by

$$\Delta S = \int_A^B dS \tag{10}$$

thus

$$\Delta S = \int_A^B (nC_V\frac{dT}{T} + nR\frac{dV}{V}) = [nC_V\ln T + nR\ln V]_A^B = nC_V\ln\frac{T_2}{T_1} + nR\ln\frac{V_2}{V_1} \tag{11}$$

For an adiabatic process, we have PV^γ =constant. Using $PV = nRT$, we have

$$V_1^{\gamma-1}T_1 = V_2^{\gamma-1}T_2 \tag{12}$$

thus

$$\ln\frac{T_2}{T_1} = \ln\frac{V_1^{\gamma-1}}{V_2^{\gamma-1}} = (\gamma-1)\ln\frac{V_1}{V_2} = \frac{C_P - C_V}{C_V}\ln\frac{V_1}{V_2} = \frac{R}{C_V}\ln\frac{V_1}{V_2} \tag{13}$$

It follows that

$$\Delta S = 0 \tag{14}$$

which makes sense since $Q = 0$ in the adiabatic process.

3) The temperature of n-mol ideal gas has changed from T_1 to T_2 degrees. Determine the entropy change for 1) under constant pressure and 2) under constant volume. (Consider reversible processes.)

Solution : Under constant pressure, heat is given by $Q = nC_P\Delta T$ or $dQ = nC_P dT$. The formulas hold equivalently for constant volume with C_V. Entropy change is then given by

$$\Delta S = \int_{T_1}^{T_2}\frac{dQ}{T} = nC_P\int_{T_1}^{T_2}\frac{dT}{T} = nC_P\ln\frac{T_2}{T_1} \tag{15}$$

Similarly for the constant volume, we obtain

$$\Delta S = \int_{T_1}^{T_2} \frac{dQ}{T} = nC_V \int_{T_1}^{T_2} \frac{dT}{T} = nC_V \ln \frac{T_2}{T_1} \tag{16}$$

Those relations can also be obtained from the entropy change for the general case

$$\Delta S = nC_V \ln \frac{T_2}{T_1} + nR \ln \frac{V_2}{V_1} \tag{17}$$

shown before. If the volume is constant, $V_1 = V_2$ and

$$\Delta S = nC_V \ln \frac{T_2}{T_1} \tag{18}$$

If the pressure is constant, from $PV = nRT$, it follows that

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = \frac{nR}{P} \tag{19}$$

and

$$\Delta S = nC_V \ln \frac{T_2}{T_1} + nR \ln \frac{V_2}{V_1} = n(C_V + R) \ln \frac{T_2}{T_1} = nC_P \ln \frac{T_2}{T_1} \tag{20}$$

4) Thermalisation of two blocks: An isolated system consists of two homogeneous metallic blocks, labeled 1 and 2, that can be considered as rigid systems (V = const). These blocks contain n_1 and n_2 moles of a metal. The blocks, initially separated, have the temperatures T_1 and T_2. When they are brought into contact, they evolve asymptotically towards a thermal equilibrium at final temperature T_f. The internal energy $E_{int,i}$ of each block (i=1,2) is a function of its temperature T_i and the number of moles n_i of metal in each block:

$$E_{int,i} = 3n_i RT_i \tag{21}$$

where R is a positive constant.

1. Determine the final temperature T_f of the system.
2. Compute the entropy variation ΔS of the system during the process that leads to its thermal equilibrium.
3. What is the entropy variation in the particular case of $n_1 = n_2 = n$?

Solution :

1. Given that the system is isolated, the internal energy is constant. Thus, the internal energy variation of the total system is:

$$\Delta E_{int} = \Delta E_{int,1} + \Delta E_{int,2} = 0 \tag{22}$$

The variation of internal energy of each black is given by:

$$\begin{aligned} \Delta E_{int,1} &= 3n_1 R \int_{T_1}^{T_f} dT = 3n_1 R(T_f - T_1) \\ \Delta E_{int,2} &= 3n_2 R \int_{T_2}^{T_f} dT = 3n_2 R(T_f - T_2) \end{aligned} \tag{23}$$

Thus, the final T_f of the system can be computed as follows:

$$\begin{aligned} &3n_1 R(T_f - T_1) = -3n_2 R(T_f - T_2) \\ &n_1 T_f - n_1 T_1 = -n_2 T_f + n_2 T_2 \\ &n_1 T_f + n_2 T_f = n_1 T_1 = n_2 T_2 \\ \rightarrow &T_f = \frac{n_1 T_1 + n_2 T_2}{n_1 + n_2} \end{aligned} \tag{24}$$

2. The entropy variation of the system is:

$$\Delta S = \Delta S_1 + \Delta S_2 \tag{25}$$

Given that the volume is constant ($dW = 0$), the internal energy variation dE_{int} is equal to dQ. Thus, the entropy variation dS of each block can be written as:

$$\begin{aligned} dS_1 &= \frac{dE_{int,1}}{T_1} = 3n_1 R \frac{dT_1}{T_1} \\ dS_2 &= \frac{dE_{int,2}}{T_2} = 3n_2 R \frac{dT_2}{T_2} \end{aligned} \tag{26}$$

Which implies that the entropy variation of each block is given by (consider all "log" in the following as "ln", as in the lecture slides):

$$\begin{aligned} \Delta S_1 &= 3n_1 R \int_{T_1}^{T_f} \frac{dT}{T} = 3n_1 R \log\left(\frac{T_f}{T_1}\right) \\ \Delta S_2 &= 3n_2 R \int_{T_2}^{T_f} \frac{dT}{T} = 3n_2 R \log\left(\frac{T_f}{T_2}\right) \end{aligned} \tag{27}$$

Thus, the entropy variation during the process towards thermal equilibrium yields:

$$\Delta S = 3n_1 R \log\left(\frac{T_f}{T_1}\right) + 3n_2 R \log\left(\frac{T_f}{T_2}\right) \tag{28}$$

3. In the particular case where $n_1 = n_2 = n$, the final temperature is given by:

$$T_f = \frac{1}{2}(T_1 + T_2) \tag{29}$$

which implies that the entropy variation is recast as:

$$\Delta S = 3nR \log\left(\frac{T_f^2}{T_1 T_2}\right) = 3nR \log\left(\frac{(T_1 + T_2)^2}{4T_1 T_2}\right) \tag{30}$$

Since the temperatures T_1 and T_2 are different, taking into account the identity:

$$(T_1 - T_2)^2 = T_1^2 + T_2^2 - 2T_1 T_2 > 0 \rightarrow T_1^2 + T_2^2 > 2T_1 T_2 \tag{31}$$

and thus

$$(T_1 + T_2)^2 = T_1^2 + T_2^2 + 2T_1 T_2 > 4T_1 T_2 \tag{32}$$

The entropy variation is strictly positive (as expected from the 2nd for an isolated system):

$$\Delta S > 3nR \log\left(\frac{4T_1 T_2}{4T_1 T_2}\right) = 0 \tag{33}$$