General Physics II: Tutorial Material

Lecture 3 on Mathematical Notes

1 Balloon

A helium-filled balloon escapes a child's hand at sea level where the atmosphere is in 1 atm and 20.0°C. When it reaches an altitude of 3600 m where the temperature is 5.0°C and the pressure 0.68 atm, how will its volume compare to that at the sea level.

Solution: We consider He gas to be an ideal gas. Since the He in the balloon does not escape, the volume, pressure and temperature at the sea level (0 m *) and those at* 3600 m *follows the relation*

$$\frac{P_0V_0}{T_0}=\frac{P_{3600}V_{3600}}{T_{3600}}$$

thus

$$\frac{V_{3600}}{V_0}=\frac{P_0T_{3600}}{P_{3600}T_0}$$

By inserting $Po = 1\ \mathrm{atm}, P_{3600} = 0.68\ \mathrm{atm}, T_0 = 273+20\ \mathrm{K}$, *and* $T_{3600} = 273+5\ \mathrm{K}$, *we obtain*

$$\frac{V_{3600}}{V_0}=\frac{1\times 278}{0.68\times 293}=1.4$$

i.e., the volume increases by 40% *at* 3600 m.

2 Air bubble

An air bubble with a diameter of 3.60 mm was created at the bottom of the lake, which is 2.5m deep. When the bubble reached the surface of the lake, where the temperature is 27°C, the diameter of the air bubble became 4.00 mm. The pressure of the atmosphere at the surface of the lake was 1 atm. What is the temperature of the water at the bottom of the lake? The density of the water is 1×10^3 kg and the gravitational acceleration constant g is 9.80 m/s^2, and assume that the air behaves as an ideal gas.

Solution:
The pressure, volume and temperature of the bubble at the surface (bottom) of the lake are denoted as $P_\mathrm{s}\,(P_\mathrm{b}), V_\mathrm{s}\,(V_\mathrm{b})$ *and* $T_\mathrm{s}\,(T_\mathrm{b})$, *respectively.*
Since the amount of air in the bubble did not change, $P_\mathrm{s}V_\mathrm{s}/T_\mathrm{s} = nR = P_\mathrm{b}V_\mathrm{b}/T_\mathrm{b}$, *thus*

$$T_\mathrm{b}=\frac{P_\mathrm{b}V_\mathrm{b}}{P_\mathrm{s}V_\mathrm{s}}T_\mathrm{s}$$

At the surface, we have $P_\mathrm{s} = 1\ \mathrm{atm} = 1.013\times10^5\ \mathrm{Pa} = 1.013\times10^5\ \mathrm{N/m^2}$

$V_\mathrm{s} = 4\pi r^3/3 = 4\pi\left(4\times10^{-3}\ \mathrm{m}/2\right)^3/3 = 3.351\times10^{-8}\ \mathrm{m}^3, r$ *is the diameter of the bubble*

$T_\mathrm{s} = 27°\mathrm{C} = 300.2°\mathrm{K}$

The pressure at the bottom of the lake is given by $P_\mathrm{b} = P_\mathrm{s} + P_{water}$ *where* P_{water} *is the pressure of the* 2.5 m *of water due to the gravitational force given by* $P_{water} = g\times\rho_{water}\times d_{water} =$

$\left(9.80 \text{ m/s}^2\right) \times \left(1 \times 10^3 \text{ kg/m}^3\right) \times (2.5 \text{ m}) = 2.45 \times 10^4 \text{ N/m}^2.$

This is because $P = F/A = (m^*a)/A$*; g relates to a for gravity from earth,* $\rho = m/V$ *and d is the height* $\rightarrow g\ \rho\ d = (m^*a)/A$*.*

It follows that

$$P_\text{b} = 1.258 \times 10^5 \text{ N/m}^2$$
$$V_\text{b} = 4\pi r^3/3 = 4\pi \left(3.6 \times 10^{-3} \text{ m}/2\right)^3/3 = 2.442 \times 10^{-8} \text{ m}^3$$

with r being the bubble diameter at the bottom

and, thus:

$$T_\text{b} = \frac{P_\text{b} V_b}{P_\text{s} V_\text{s}} T_\text{s} = 272°\text{K}$$

3 Hydrogen in outer space

In our outer space, the density of matter is 1 atom per cm^3. It is dominated by the hydrogen atom and at a temperature of 2.7 K. What is the rms-speed of those hydrogen atoms? What is the pressure there in the unit of atm?

Solution:
The mass of the hydrogen atom is $1\text{u} = 1.66 \times 10^{-27}$ kg*, and the number of the hydrogen atom in the universe per* m^3 *is* 10^6*. From the Boltzmann's equation, the rmsspeed is given by*

$$v_\text{rms} = \sqrt{3\frac{\text{kT}}{\text{m}}} = \sqrt{3\frac{1.38 \times 10^{-23} \text{ J/K} \times 2.7 \text{ K}}{1.66 \times 10^{-27} \text{ kg}}} = 259 \text{ m/s}$$

The pressure is given by the ideal gas law as

$$P = \frac{NkT}{V} = \frac{10^6 \times 1.38 \times 10^{-23} \text{ J/K} \times 2.7 \text{ K}}{1 \text{ m}^3} = 3.726 \times 10^{-17} \text{ Pa}$$

Using the conversion factor, $1 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$*, we obtain*

$$P = 3.726 \times 10^{-17} \text{ Pa} = \frac{3.726 \times 10^{-17}}{1.01 \times 10^5} = 3.7 \times 10^{-22} \text{ atm}$$

4 Vacuum pressure

The lowest pressure attainable using the best available vacuum technique is about $10^{-12} N/m^2$. At such a pressure, how many molecules are there per cm^3 at 0°C?

Solution: Assuming the ideal gas law is valid, the number of molecules is given by

$$N = \frac{PV}{kT}$$

where k is the Boltzmann constant, $= 1.38 \times 10^{-23} \text{ J/K} = 1.38 \times 10^{-23} \text{Nm/K}$*. For* $P = 10^{-12} \text{Nm}^{-2}, V = 1 \text{ cm}^3 = 10^{-6} \text{ m}^3$*, and* $T = 273$ K*, we obtain the number of molecules per cubic cm to be*

$$N = \frac{10^{-12} \text{Nm}^{-2} \times 10^{-6} \text{ m}^3}{1.38 \times 10^{-23} \text{NmK}^{-1} \times 273 \text{ K}} = 265$$

5 rms-speed of molecules

Show that the rms-speed of gas molecules is given by $v_{rms} = \sqrt{3P/\rho}$, where P and ρ are the pressure and density of the gas respectively.

Solution:
From the Maxwell's distribution, the rms-velocity is given by

$$v_{\rm rms} = \sqrt{3\frac{kT}{m}}$$

where k is the Boltzmann's constant and m is the mass of the gas molecule. The ideal gas law in terms of the Boltzmann's constant is given by $PV = NkT$ where N is the number of molecules. We can replace kT with PV/N and thus, it follows that

$$\frac{kT}{m} = \frac{PV}{Nm} = \frac{P}{Nm/V} = \frac{P}{\rho}$$

where $\rho \equiv Nm/V$ is the density of the gas. It follows that

$$v_{\rm rms} = \sqrt{3\frac{kT}{m}} = \sqrt{3\frac{P}{\rho}}$$

6 The bicycle pump

A bicycle pump takes a volume ΔV of air at atmospheric pressure P_0 and constant temperature T_0 and compresses it so that it enters a tyre that has a volume V_0. The air inside the tyre is initially at atmospheric pressure P_0 and can be considered as an ideal gas. Determine the number of times N the user has to pump air into the tyre to reach a pressure P_f. Assume that the pump is designed such that the air in the tyre is always at temperature T_0.
Compute N for $V_0 = 50$ l, $\Delta V = 1.2$ l, and $P_f = 2.5 P_0$.

Solution: The initial and final number of moles of air inside the tyre of volume V_0 at temperature T_0 are given by:

$$n_0 = \frac{P_0 V_0}{RT_0} \text{ and } n_f = \frac{P_f V_0}{RT_0} \rightarrow \frac{n_f}{n_0} = \frac{P_f}{P_0}$$

The additional number of moles of air pumped into the tyre each time are and the final number of moles are:

$$\Delta n = \frac{P_0 \Delta V}{RT_0} \text{ and } n_f = n_0 + N\Delta n$$

Here, N is the number of pumps.
Dividing the last equation by n_0 and replacing Δn with the above expression, we obtain:

$$\frac{n_f}{n_0} = 1 + N\frac{\Delta n}{n_0} = 1 + N\frac{P_0 \Delta V}{n_0 RT_0} = 1 + N\frac{\Delta V}{V_0} = \frac{P_f}{P_0}$$

which implies that (by solving the equation for N):

$$N = \left(\frac{P_f}{P_0} - 1\right)\frac{V_0}{\Delta V} = 62.5$$

This means that the air has to be pumped 63 times in order to reach a final pressure that is at least 2.5 times the initial pressure.

7 Basic mathematical concepts: Differentials of state functions

NOTE: Some of you may not be yet familiar with partial differentials – then please ignore this exercise until you have it in your math course, or you ask your TA!

7.1 Differentials of state functions

Consider the function $f(x, y) = y \exp(ax) + xy + bx \ln y$ where a and b are constants.
a) Calculate $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, and $df(x, y)$
b) Calculate $\frac{\partial^2 f(x,y)}{\partial x \partial y}$

Solution:
a)

$$\frac{\partial f(x,y)}{\partial x} = ay \exp(ax) + y + b \ln y$$
$$\frac{\partial f(x,y)}{\partial y} = \exp(ax) + x + \frac{bx}{y}$$

$$df(x,y) = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy = (ay \exp(ax) + y + b \ln y) dx + \left(\exp(ax) + x + \frac{bx}{y} \right) dy$$

b)

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial (ay \exp(ax) + y + b \ln y)}{\partial y} = a \exp(ax) + 1 + \frac{b}{y}$$

Note that it doesn't matter whether you differentiate first with respect to x and then to y or first with respect to y and then to x.

7.2 Differentials of the ideal gas equation

Consider the equation of state for ideal gas for the following tasks (PV = nRT).
a) Calculate the differential dP(T,V)
b) Calculate

$$\frac{\partial}{\partial T} \left(\frac{\partial P(T,V)}{\partial V} \right) \text{ and } \frac{\partial}{\partial V} \left(\frac{\partial P(T,V)}{\partial T} \right)$$

c) Cyclic rule for ideal gas: Calculate

$$\frac{\partial P(T,V)}{\partial T} \frac{\partial T(P,V)}{\partial V} \frac{\partial V(T,P)}{\partial P}$$

Solution:
a)

$$dP(T,V) = \frac{nR}{V} dT - \frac{nRT}{V^2} dV$$

b)

$$\frac{\partial}{\partial T} \left(\frac{\partial P(T,V)}{\partial V} \right) = \frac{\partial}{\partial V} \left(\frac{\partial P(T,V)}{\partial T} \right) = -\frac{nR}{V^2}$$

c) The partial derivatives of the pressure, the temperature and the volume of an ideal gas (PV=nRT) are given by:

$$\frac{\partial P(T,V)}{\partial T} = \frac{\partial}{\partial T} \left(\frac{nRT}{V} \right) = \frac{nR}{V}$$
$$\frac{\partial T(P,V)}{\partial V} = \frac{\partial}{\partial V} \left(\frac{PV}{nR} \right) = \frac{P}{nR}$$
$$\frac{\partial V(T,P)}{\partial P} = \frac{\partial}{\partial P} \left(\frac{nrT}{P} \right) = -\frac{nR}{P^2} = -\frac{V}{PT}$$

Thus:

$$\frac{\partial P(T,V)}{\partial T}\frac{\partial T(P,V)}{\partial V}\frac{\partial V(T,P)}{\partial P} = -1$$

We will see later that this rule can be generalised.

7.3 State Function: Rubber Cord

A rubber cord of length L, which can be described as a state function L(T, F) of the temperature T of the cord and of the forces of magnitude F applied at each end to stretch it. The two physical properties of the cord are:

a) the Young modulus, which can be expressed as $E = \frac{L}{A}\left(\frac{\partial L}{\partial F}\right)^{-1}$ with A being the cross section area. Note that this is the same (but mathematical) definition as in the lecture notes only for small changes and solved for E.

b) the thermal expansion coefficient α, which can be expressed as $\alpha = \frac{1}{L}\frac{\partial L}{\partial T}$.

Determine how much the length of the cord varies if its temperature changes by dT and at the same time the force F changes by dF. Assume that $dT << T$ and $dF << F$. Express dL in terms of E and α.

Solution:

According to the definition of a differential, we can express the change of length of the rubber cord dL as:

$$dL = \frac{\partial L}{\partial T}dT + \frac{\partial L}{\partial F}dF$$

which can be recast as:

$$dL = L\left(\frac{1}{L}\frac{\partial L}{\partial T}\right)dT + \frac{L}{A}\left(\frac{L}{A}\left(\frac{\partial L}{\partial F}\right)^{-1}\right)^{-1} dF$$

Using the two physical properties of the cord, we obtain an expression for the change of the length of the rubber cord:

$$dL = L\alpha dT + \frac{L}{AE}dF$$

Comment: This is a mathematically elegant way to describe at the same time the expansion of the rubber cord due to a change in temperature and due to a tensile stress as a consequence of a force/pressure change at the ends of the rubber cords.