General Physics II: Tutorial Material

Lecture 12 (Mock Exam II)

1) Let us consider a rigid and hermetic room with a volume of 22.6 m^3 and filled with air with a pressure of 1 atm at 0°C. We assume that air is an ideal gas. In the following, assume that $0°\mathrm{C} = 273$ K and 1 atm $= 1.013 \times 10^5$ $\mathrm{N/m}^2$, and for the air, the molecular specific heat $C_P = 7$ cal/(mol · K) and $\gamma = C_P/C_V = 1.4$ are assumed to be constant. Another useful number is 1 Nm = 1 J $= 2.39 \times 10^{-4}$ kcal.

a) The air temperature in the room is quasi-statically increased from 0° to 20°C. Draw the P-V (pressure versus volume) diagram of this process and calculate the heat, Q, necessary for this process.

b) Fixture of one of the walls is removed and the wall can move freely such that the pressure of the air inside can be fixed to 1 atm. However, it is still hermetic so that the air inside cannot leak outside. How large is the heat, Q, needed to quasi-statically increase the air temperature of the room from 0° to 20°C?

c) Instead of removing the wall fixture, we make a very small hole on the wall, which allows the air in the room to escape outside very slowly if required, such that the pressure of the air inside the room is always kept at 1 atm. How large is the heat, Q, needed to quasi-statically increase the air temperature of the room to 20°C?

Solution:

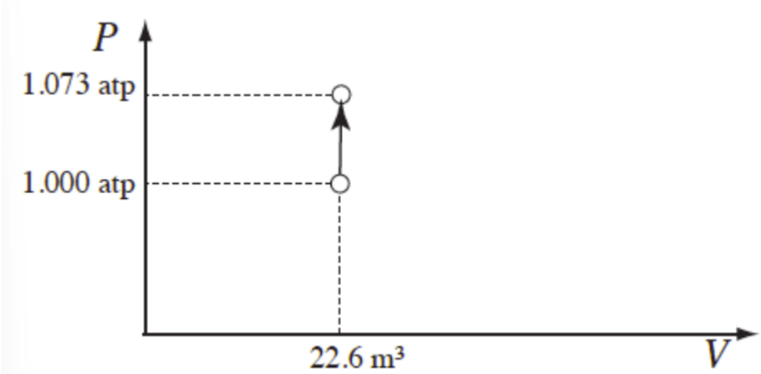
a) From the values of γ and C_P given in the problem, the universal gas constant can be obtained as $R = C_P - C_V = C_P(1 - 1/\gamma) = 2$ cal/(mol · K). The mole number, n, for the air in the room is given by the ideal gas law as

$$n = \frac{PV}{RT} \approx 1 \times 10^3 \text{ mole} \tag{1}$$

where $P = 1$ atm $= 1.013 \times 10^5$ $\mathrm{N/m}^2$, $V = 22.6$ m^3 and $T = 273$ K. Since the room is hermetic and rigid, the volume and amount of the air does not change, thus from the ideal gas law P/T is constant. Therefore, the pressure, P, of the air in the room at the temperature $T = 20°\mathrm{C} = 293$ K is given by

$$P = 1 \text{ atp} \times \frac{293 \text{ K}}{273 \text{ K}} \approx 1.073 \text{ atp} \tag{2}$$

Hence the P-V diagram of the process is given by



Since it is a process under the constant volume, the necessary heat is given by

$$Q = \int_{T_0}^{T_1} nC_V dT = nC_V \int_{T_0}^{T_1} dT = nC_V \Delta T \tag{3}$$

where $\Delta T = 20^\circ$ K and

$$C_V = C_P/\gamma = 7/1.4 = 5cal/(mol \cdot K) \tag{4}$$

It follows that the necessary heat is given by

$$Q = nC_V \Delta T = 100kcal \tag{5}$$

b) Since the pressure of the air, P, is kept constant and the air stays inside of the room, the necessary thermal energy to increase the temperature by ΔT is given by

$$Q = nC_P \Delta T = 140 \text{ kcal}. \tag{6}$$

c) When the air temperature increases, part of the air escapes through the hole, thus the air has to **make work**, and **the pressure, P, remains constant**. Therefore, the amount of the air, n, changes in the process. Also the temperature increase is quasi-static and the ideal gas law is valid for the air inside of the room and

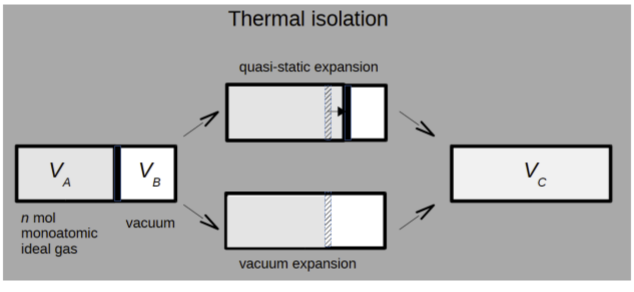
$$n = \frac{P_0 V_0}{RT} \tag{7}$$

where $P_0 = 1$ atm $= 1.013 \times 10^5$ N/m^2 and $V_0 = 22.6$ m^3, i.e. n is a function of T. Therefore, the thermal energy needed to increase the air temperature from T_0 to T_i is given by,

$$Q = \int_{T_0}^{T_1} C_P n dT = C_P \int_{T_0}^{T_1} \frac{P_0 V_0}{RT} dT = \frac{P_0 V_0}{R} C_P \ln \frac{T_1}{T_0}. \tag{8}$$

$P_0 = 1$ atm $= 1.013 \times 10^5$ N/m^2, $V_0 = 22.6$ m^3, $T_0 = 273$ K, $T_1 = 293$ K and $R = 2$ cal/(mole $\cdot$ K) lead to the equation to $Q = 135$ kcal.

2) We consider a system of a hermetic cylinder with a volume V_C placed in a thermally isolated environment. The cylinder is split into two volumes, V_A and V_B, with a hermetic wall. The volume V_A is filled with an n mol ideal gas with a pressure P_A and volume V_B kept in vacuum.



a) The wall is moved quasi-statically so that the volume V_A becomes V_C.

- i. What are the pressure and temperature of the gas when it reaches to the volume V_C?
- ii. Calculate the entropy change, ΔS, for this process.

b) Instead of moving the wall quasi-statically, it is removed suddenly to let the gas make vacuum expansion for its volume to become V_C, which is called "free expansion". After the system reaching equilibrium:

- i. What is the pressure and temperature of the system for the final state?
- ii. Calculate the entropy change for this process.

Solution a):

Using the ideal gas law, the temperature of the initial state is given by

$$T_A = \frac{V_A P_A}{nR} \tag{9}$$

- i. Since the system in thermal isolation, there is no heat transfer (but work is done by the system). Therefore, the process is an adiabatic expansion of the system to the volume V_C, i.e. $P_A V_A^\gamma = P_C V_C^\gamma$, where P_C is the pressure of the final state and $\gamma = C_P/C_V$, thus

$$P_C = P_A \left(\frac{V_A}{V_C}\right)^\gamma \tag{10}$$

  And using the ideal gas law, the temperature of the final state is then given by

$$T_C = \frac{V_C P_C}{nR} = \frac{V_C P_A}{nR}\left(\frac{V_A}{V_C}\right)^\gamma \tag{11}$$

- ii. This process is reversible where entropy difference is solely given by

$$\Delta S = \int_{\text{initial}}^{\text{final}} \frac{dQ}{T} = 0 \tag{12}$$

  since there is no heat exchange with the environment (and no entropy production).

Solution b):

- i. Since the gas expands without pressure, there is no work in vacuum expansion (total volume of the system stays the same). As the system is in thermal isolation, there is also no heat exchange with the environment. Therefore, the internal energy of the system is identical to the initial state after the expansion, i.e. the temperature remains unchanged after the expansion. The final state after the vacuum expansion has volume V_C, and the temperature identical to the initial state

$$T_C = T_A = \frac{V_A P_A}{nR} \tag{13}$$

  Once the system reaches its equilibrium, the ideal gas law can be used and the pressure of the final state is given by:

$$P = \frac{nRT_A}{V_C} \tag{14}$$

- ii. The process of free vacuum expansion is not a reversible process, thus

  $$\Delta S = \int_{\text{initial}}^{\text{final}} \frac{dQ}{T} \tag{15}$$

  cannot be used to calculate the entropy change, as there is a contribution from entropy production within the system (see lecture notes). On the other hand, entropy depends only on the initial state and final state, but not on the path in between. Therefore, a reducible process reaching to the same final state can be used to calculate the entropy difference using the above equation. As T stays the same, isothermal expansion can bring the initial state to this final state. On the isothermal path, there is no change in the internal energy, thus, $W = Q$. The entropy change can be then computed by

  $$\Delta S = \int_{\text{initial}}^{\text{final}} \frac{dQ}{T} = \frac{1}{T_A} \int_{V_A}^{V_C} P dV = nR \int_{V_A}^{V_C} \frac{dV}{V} = nR \ln \frac{V_C}{V_A} \tag{16}$$

Clarification for a) ii:
To formalize why the temperatures T_A and T_C are equal, we can make use of thermodynamic subsystems. One considers the system to be the whole cylinder, and the parts V_A (labeled A) and V_B (labeled B) subsystems of it. Then, for the full system, since $\delta Q = 0$ (thermal isolation) and $\delta W = 0$ (fixed volume), the internal energy doesn't change by virtue of the second law, $dU = 0$. Since U is an extensive state variable, $U = U_A + U_B$. But B contains a vacuum, and thus no internal energy, such that $U \equiv U_A$. Given that $dU = 0$, we know that the internal energy of the final configuration is $U_C = U = U_A$. Since the internal energy of an ideal gas depends only on temperature, this implies $T_C = T_A$.

3) Gas molecules are moving randomly due to thermal motion. Using the Boltzmann factor, $e^{-E/kT}$, where E is the kinetic energy, k is the Boltzmann constant and T is the temperature of the gas, probability for a monoatomic gas molecule has three vector-velocity with a value between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$, is given by

$$P(v_x, v_y, v_z) dv_x dv_y dv_z = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m}{2kT}(v_x^2 + v_y^2 + v_z^2)\right) dv_x dv_y dv_z. \tag{17}$$

a) What is the probability for a molecule to have a velocity in z-direction between v_z and $v_z + dv_z$, while it can have any values in x and y direction?

b) What is the average velocity in z for the atoms moving toward the positive z direction? What is the average velocity in z for the atoms moving toward the negative z direction?

c) Would these two velocities be different if the gas molecule were diatomic? Why?

Solution:

a) Since v_x and v_y can take any values, the desired probability for v_z, $P(v_z)dv_z$, is given by integrating the $P(v_x, v_y, v_z)dv_x dv_y dv_z$ over v_x and v_y from $-\infty$ to $+\infty$:

$$\begin{aligned}
P(v_z)dv_z &= \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y P(v_x, v_y, v_z)dv_z \\
&= \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}\right) dv_z \\
&= \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \left[\int_{-\infty}^{\infty} \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_x^2}{2kT}\right) dv_x\right] \\
&\quad \times \left[\int_{-\infty}^{\infty} \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_y^2}{2kT}\right) dv_y\right] \\
&= \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \times 1 \times 1 \\
&= \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \qquad (18)
\end{aligned}$$

b) If we consider only positive v_z, the probability distribution $P(v_z)$ is not normalized to 1 because

$$\int_0^{\infty} P(v_z)dv_z = \frac{1}{2}\int_{-\infty}^{\infty} P(v_z)dv_z = \frac{1}{2} \qquad (19)$$

Therefore, this normalization factor needs to be taken into account when calculating $\langle v_z\rangle_+$, which is given by

$$\begin{aligned}
\langle v_z\rangle_+ &= \frac{\int_0^{\infty} v_z P(v_z)dv_z}{\int_0^{\infty} P(v_z)dv_z} \\
&= 2\int_0^{\infty} v_z P(v_z)dv_z \\
&= 2\int_0^{\infty} v_z \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \\
&= 2\left(\frac{m}{2\pi kT}\right)^{1/2} \left[-\frac{kT}{m}\exp\left(-\frac{mv_z^2}{2kT}\right)\right]_0^{\infty} \\
&= 2\left(\frac{m}{2\pi kT}\right)^{1/2} \left(0 - \left(-\frac{kT}{m}\right)\right) \\
&= \left(\frac{2kT}{\pi m}\right)^{1/2} \qquad (20)
\end{aligned}$$

Similarly, the average value of v_z for $v_z < 0$, $\langle v_z\rangle_-$, is given by

$$\begin{aligned}
\langle v_z\rangle_- &= \frac{\int_{-\infty}^{0} v_z P(v_z)dv_z}{\int_{-\infty}^{0} P(v_z)dv_z} \\
&= 2\int_{-\infty}^{0} v_z P(v_z)dv_z \\
&= 2\int_{-\infty}^{0} v_z \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z \\
&= 2\left(\frac{m}{2\pi kT}\right)^{1/2} \left[-\frac{kT}{m}\exp\left(-\frac{mv_z^2}{2kT}\right)\right]_{-\infty}^{0} \\
&= 2\left(\frac{m}{2\pi kT}\right)^{1/2} \left(-\frac{kT}{m} - 0\right) \\
&= -\left(\frac{2kT}{\pi m}\right)^{1/2} \qquad (21)
\end{aligned}$$

c) The kinetic energy of diatomic molecule has additional terms due to rotations. However, once they are integrated out in the probability distribution, as done for v_x and v_y, the remaining probability distribution for v_z is identical to that of monoatomic molecule. Therefore, two velocities are identical to those for the monoatomic molecule.