General Physics II: Tutorial Material

Lecture 10 (Chapter 9, Thermal machines)

1) Braking cycle
A system is made up of a vertical cylinder which is sealed at the top and closed by a piston at the bottom. A valve A controls the intake of gas at the top and an exhaust valve B (also at the top) is held back by a spring that exerts a constant pressure P2 on the valve. The system goes through the following processes:

- $0 \to 1$: the piston is at the top of the cylinder; Valve A opens up and the piston is lowered into it so that some of the gas at atmospheric pressure $P_0 = P_1$ is added to the cylinder. The gas is at room temperature T_1. Valve B is closed. The maximum volume occupied by the incoming gas is V_1.
- $1 \to 2$: Valve A is now closed and the piston moves upward, fast enough so that the process can be considered adiabatic. Valve B remains closed as long as the pressure during the rise of the piston is lower than P_2. As the piston continues its rise, the gas reaches pressure $P_2 = 10P_1$, at a temperature T_2 in a volume V_2. Assume a reversible adiabatic process.
- $2 \to 3$: As the piston keeps moving up, valve B opens up, the pressure is $P_3 = P_2$ and the gas is released in the environment while valve A still remains closed until the piston reaches the top, where $V_3 = V_0 = 0$.
- $3 \to 0$: Valve B closes and valve A opens up. The system is ready to start over again.

Analyze this cycle by using the following instructions:

a) Draw the P-V diagram for the 3 processes that the system is undergoing.

b) Determine the temperature T_2 and volume V_2.

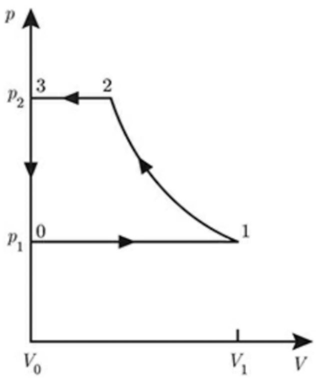
c) Find the work W performed per cycle.

Numerical application:

$$V_0 = V_3 = 0,\ P_0 = P_1 = 1 \times 10^5 \text{ Pa},\ V_1 = 0.25 \text{ L},\ T_1 = 27°\text{C},\ \gamma = 1.4 \tag{1}$$

Solution:

a) The P-V diagram consists of an isobaric expansion ($0 \to 1$), an adiabatic compression ($1 \to 2$), an isobaric contraction ($2 \to 3$) and an isochoric decompression ($3 \to 0$).



b) For the adiabatic compression, from the adiabatic equations we obtain:

$$T_1^{\gamma} P_1^{1-\gamma} = T_2^{\gamma} P_2^{1-\gamma} \tag{2}$$

which implies that

$$T_2 = T_1 \left(\frac{P_1}{P_2}\right)^{(1-\gamma)/\gamma} = T_1 \left(\frac{1}{10}\right)^{(1-\gamma)/\gamma} = 579K \tag{3}$$

The first adiabatic equation we derived in the lecture is:

$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma} \tag{4}$$

which implies that

$$V_2 = V_1 \left(\frac{P_1}{P_2}\right)^{1/\gamma} = V_1 \left(\frac{1}{10}\right)^{1/\gamma} = 0.048l \tag{5}$$

c) The work performed on the gas over the entire cycle is the sum of the works performed during the four processes:

$$W = W_{01} + W_{12} + W_{23} + W_{30} \tag{6}$$

The work performed during the isobaric process is:

$$W_{01} + W_{23} = P_1 \int_0^{V_1} dV + P_2 \int_{V_2}^{0} dV = P_1 V_1 - P_2 V_2 \tag{7}$$

There is no work performed during the isochoric process: $W_{30} = 0$. The work performed during the adiabatic process is:

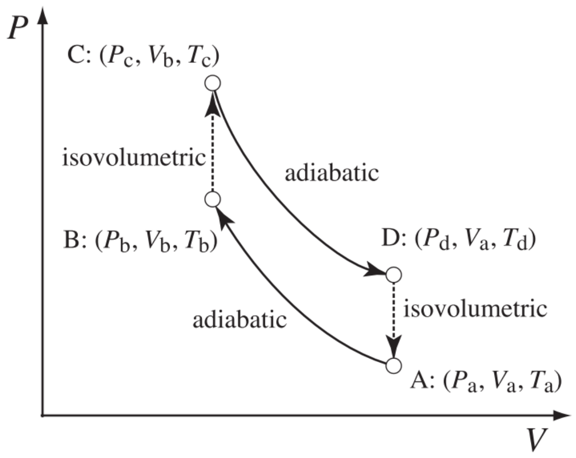
$$W_{12} = \int_{V_1}^{V_2} P dV = P_1 V_1^{\gamma} \int_{V_1}^{V_2} \frac{dV}{V^{\gamma}} = \frac{P_1 V_1^{\gamma}}{1-\gamma}(V_2^{1-\gamma} - V_1^{1-\gamma}) = \frac{1}{1-\gamma}(P_2 V_2 - P_1 V_1) \tag{8}$$

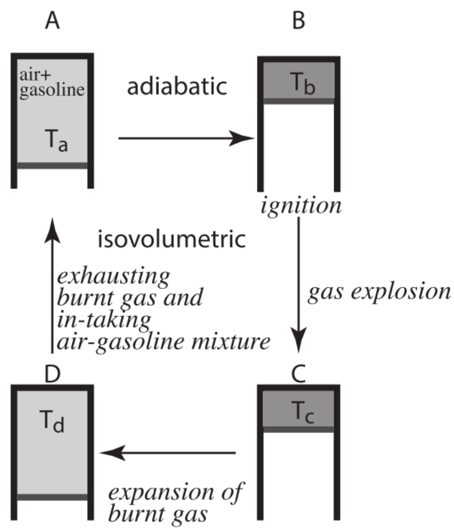
Thus,

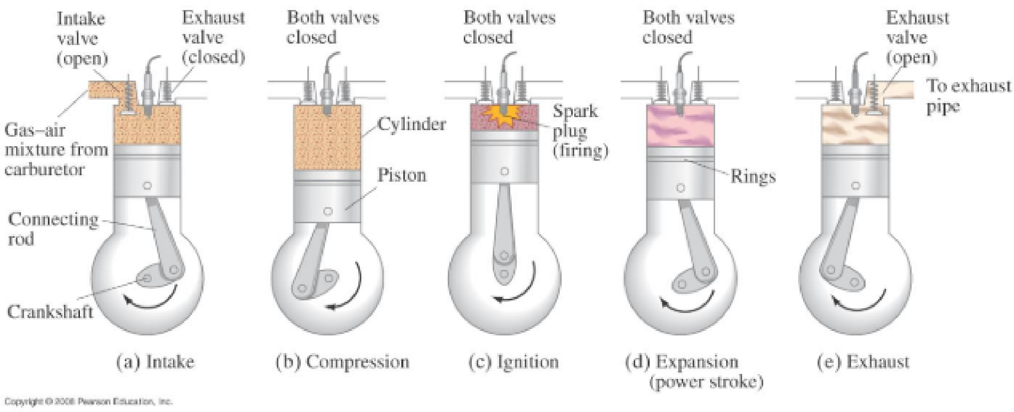
$$W = \frac{\gamma}{\gamma - 1}(P_1 V_1 - P_2 V_2) = \frac{\gamma}{\gamma - 1} P_1 (V_1 - 10 V_2) = -80.5kJ \tag{9}$$

Since the work is negative (remember $\Delta E_{int} = Q - W$), work is done on the system, i.e. the system can act as a brake for whatever mechanism drives the motion of the piston.

2) For the Otto Cycle shown in the figures below, calculate the efficiency of the Otto cycle engine and compare with that of the Carnot cycle engine, $\epsilon_{Carnot} = 1 - \frac{T_a}{T_c}$, where T_a and T_c are the lowest and highest temperature of the system, respectively. Which one of the two engines is more efficient?







Solution:

The Otto cycle consists of four paths combining the two adiabatic and two isovolumetric paths.

$A(V_a, P_a, T_a)$: Gas (mixture of gasoline with air) in the cylinder.

$A \rightarrow B$: Adiabatic compression of gas (Q = 0, V decreases, P increases, T increases) by the movement of the piston (W < 0).

$B(V_b, P_b, T_b)$: Ignition with a spark plug (gasoline) or self-ignition (diesel)

$B \rightarrow C$: Q_H generated, P and T increase at the constant volume.

$C(V_c, P_c, T_c)$: Pressure reaches its highest point.

$C \rightarrow D$: Adiabatic expansion of the gas (Q = 0, V increases, P decreases, T decreases) by pushing down the piston (W > 0).

$D(V_c, P_c, T_c)$: The volume is at its maximum.

$D \to A$: Q_L to the environment at the constant volume. The burned gas is replaced by the new gas.

The heat in $B \to C$ is given by :

$$Q_1 = nC_V(T_c - T_b) > 0, \tag{10}$$

and similarly for $D \to A$,

$$Q_2 = nC_V(T_a - T_d) < 0. \tag{11}$$

From the first law of thermodynamics, $\Delta E_{int} = Q - W$, the work in one cycle is given by :

$$W = Q = Q_1 + Q_2. \tag{12}$$

Thus, the efficiency :

$$\epsilon = \frac{W}{Q_1} = 1 + \frac{Q_2}{Q_1} = 1 + \frac{T_a - T_d}{T_c - T_b}. \tag{13}$$

In the adiabatic processes, we have $P_a V_a^\gamma = P_b V_b^\gamma$ and $P_c V_b^\gamma = P_d V_a^\gamma$. Using the ideal gas law,

$$\frac{P_a V_a}{T_a} = \frac{P_b V_b}{T_b} \text{and} \frac{P_c V_b}{T_c} = \frac{P_d V_a}{T_d} \tag{14}$$

It follows that

$$\begin{aligned} &P_a V_a^\gamma = P_b V_b^\gamma \\ &P_a V_a^\gamma \frac{T_a}{P_a V_a} = P_b V_b^\gamma \frac{T_b}{P_b V_b} \\ &T_a V_a^{\gamma-1} = T_b V_b^{\gamma-1} \end{aligned} \tag{15}$$

and

$$\begin{aligned} &P_c V_b^\gamma = P_d V_a^\gamma \\ &P_c V_b^\gamma \frac{T_c}{P_c V_b} = P_d V_a^\gamma \frac{T_d}{P_d V_a} \\ &T_c V_b^{\gamma-1} = T_d V_a^{\gamma-1} \end{aligned} \tag{16}$$

giving

$$T_a - T_d = T_b \left(\frac{V_b}{V_a}\right)^{\gamma-1} - T_c \left(\frac{V_b}{V_a}\right)^{\gamma-1} = \left(\frac{V_b}{V_a}\right)^{\gamma-1} (T_b - T_c) \tag{17}$$

It follows that

$$\epsilon = 1 + \frac{T_a - T_d}{T_c - T_b} = 1 + \left(\frac{V_b}{V_a}\right)^{\gamma-1} \frac{T_b - T_c}{T_c - T_b} = 1 - \left(\frac{V_a}{V_b}\right)^{1-\gamma} \tag{18}$$

i.e., the efficiency is a function of the compression ratio V_b/V_a. Since $1 - \gamma < 0$, the efficiency is higher for an engine with a higher compression ratio.

From the adiabatic relation for an ideal gas, $P_a V_a^\gamma = P_b V_b^\gamma$, we obtain $(V_a/V_b)^\gamma = P_b/P_a$. The equation of the states for an ideal gas, $PV = nRT$, leads to $P_a V_a/T_a = P_b V_b/T_b$, thus we have $P_b/P_a = (V_a T_b)/(V_b T_a)$. It follows that

$$\left(\frac{V_a}{V_b}\right)^\gamma = \frac{P_b}{P_a} = \frac{V_a}{V_b}\frac{T_b}{T_a} \quad \to \quad \left(\frac{V_a}{V_b}\right)^{1-\gamma} = \frac{T_a}{T_b} \tag{19}$$

The efficiency can now be written as

$$\epsilon_{Otto} = 1 - \frac{T_a}{T_b} \tag{20}$$

By noting that in the isovolumetric process, $B \to C$, we have $P_c > P_b$, thus $T_c = T_b(P_c/P_b) > T_b$ and $T_a/T_b > T_a/T_c$. It follows that

$$\epsilon_{Otto} = 1 - \frac{T_a}{T_b} < 1 - \frac{T_a}{T_c} = \epsilon_{Carnot} \tag{21}$$

and the Otto engine is less efficient than the Carnot engine.

3) In the Otto cycle, the volume ratio in the expansion, $C \to D$, is identical to that for the compression, $A \to B$, and is given by V_a/V_b. Some hybrid cars use Atkinson cycle where the volume ratios are different. This is realised by changing the timing of exhaust or/and intake and also called Miller cycle. In order to compare its performance with the Otto cycle, we consider the Miller cycle to use the same volume of air-gasoline mixture gas, V_a, for the adiabatic compression and the condition for ignition, i.e. the adiabatic compression of the air-gasoline mixture starts at $A(P_a, V_a, T_a)$.

The Miller cycle shown in the P-V plot below is the following:

i) At A, the piston is somewhere in the middle of the cylinder. The volume, V_a, is filled with the air-gasoline mixture and all the valves are closed.

ii) The piston moves up to the top (B) and the gas is ignited and explodes ($B \to C$).

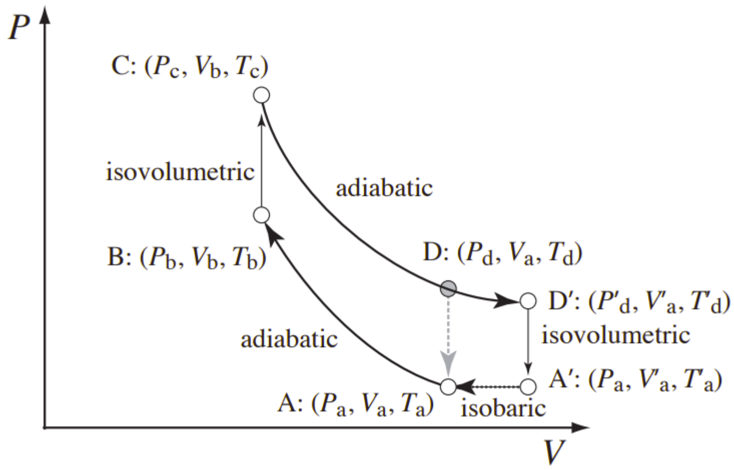
iii) The piston is pushed down ($C \to D'$) and reaches the lowest position of the cylinder (D').

iv) The piston goes up to exhaust the burnt gas (exhaust valve open) and goes down to take in the air-gasoline mixture (intake valve open), which corresponds to the isovolumetric reduction of the pressure, $D' \to A'$, where at A' the cylinder is back at the lowest position.

v) When the piston starts to move up, the intake valve is still open, thus isobaric compression starts till arriving at A where the in-take valve closes.

vi) Back to the original state and ready for the next cycle.

Figure below is the P-V plot for a Miller cycle, together with an equivalent Otto cycle.



Exercise 3.png

Show that an engine with Miller cycle ($A \to B \to C \to D' \to A' \to A$) is more efficient than that with Otto cycle ($A \to B \to C \to D \to A$).

Solution:

The figure above shows the P-V plots for the Miller cycle ($A \to B \to C \to D' \to A' \to A$) and Otto cycle ($A \to B \to C \to D \to A$) to be compared. Since the initial volume and pressure of the airgasoline mixtures are same, we assume that the two cycles start with the same amount of gas molecule from A.

As discussed previously, the area surrounded by a cycle on the P-V plane gives the total work. Therefore, the total work for the Otto cycle, W^{Otto}_{total}, is less than that of the Miller cycle, W^{Miller}_{total}, i.e. $W^{Otto}_{total} < W^{Miller}_{total}$.

Now we consider the entropy. Entropy changes for $A \to B$, $C \to D$ and $C \to D'$ are zero, for $B \to C$, $D \to A$ and $D' \to A'$

$$\Delta S_{bc} = nC_V \ln \frac{T_c}{T_b}, \Delta S_{da} = nC_V \ln \frac{T_a}{T_d}, \Delta S'_{da} = nC_V \ln \frac{T'_a}{T'_d} \tag{22}$$

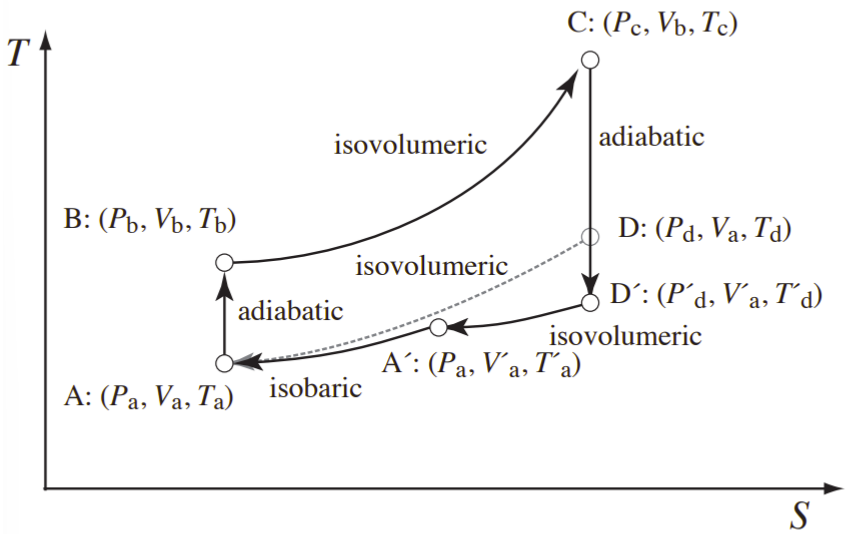
and for $A' \to A$,

$$\Delta S_{a'a} = nC_P \ln \frac{T_a}{T'_a} \tag{23}$$

Since $P_b < P_c$, and from $PV = nRT$ we have $T_b < T_c$, thus $\Delta S_{bc} > 0$.

Similarly, from $P_a < P_d$, $P'_a < P'_d$ and $V_a < V'_d$ we obtain $T_a < T_d$, $T'_a < T'_d$ and $T_a < T'_a$, thus $\Delta S_{da} < 0$, $\Delta S'_{da} < 0$ and $\Delta S_{a'a} < 0$.

PV^γ = constant in the adiabatic process and $PV = nRT$ lead to $T_a < T_b$ and $T_d < T_c$. It follows that the $S - T$ plots for the Otto cycle ($A \to B \to C \to D \to A$) and Miller cycle ($A \to B \to C \to D' \to A' \to A$) can be drawn as



Exercise 3-2.png

The thermal energy flow into the engine, Q_{in}, is the positive heat in the system given by the area below the $B \to C$ line for both Otto and Miller cycles, thus $Q_{in}^{Otto} = Q_{in}^{Miller}$. The efficiency of an engine is given by

$$\epsilon = \frac{W_{total}}{Q_{in}} \tag{24}$$

From $W_{total}^{Otto} < W_{total}^{Miller}$ and $Q_{in}^{Otto} = Q_{in}^{Miller}$ follows that the Miller cycle is more efficient.

4) A heat pump is used to warm up a room at temperature T_1 by transferring thermal energy from outside at temperature T_2, where $T_1 > T_2$ i.e. the outside is colder than the room, using work done to the heat pump. Show that a heat pump is more economical than heating the room directly with the work by computing the efficiency ("Coefficient of Performance") of the heat pump using the Carnot cycle.

Solution:

If we operate the Carnot cycle in the reversed order, $A \to D \to C \to B \to A$, heats for $D \to C$ and $B \to A$, Q_{dc} and Q_{ba}, respectively, are given by

$$Q_{dc} = nRT_2 \ln \frac{V_c}{V_d} > 0 \quad \text{and} \quad Q_{ba} = nRT_1 \ln \frac{V_a}{V_b} < 0 \tag{25}$$

and similarly for the work, $A \to D$, $D \to C$, $C \to B$, and $B \to A$

$$\begin{aligned} W_{ad} &= \frac{P_b V_b}{1-\gamma}\left[\left(\frac{V_c}{V_b}\right)^{1-\gamma} - 1\right], W_{dc} = nRT_2 \ln\frac{V_c}{V_d}, \\ W_{cb} &= \frac{P_b V_b}{1-\gamma}\left[1 - \left(\frac{V_c}{V_b}\right)^{1-\gamma}\right], W_{ba} = nRT_1 \ln\frac{V_d}{V_c} \end{aligned} \tag{26}$$

The total work is then given by

$$W_{total} = W_{ad} + W_{dc} + W_{cb} + W_{ba} = nR(T_2 - T_1)\ln\frac{V_b}{V_a} \tag{27}$$

The efficiency of a heat pump given as

$$\epsilon_{\text{heat pump}} = \frac{\text{thermal energy given to the heat reservoir with } T = T1}{\text{total work given to the heat pump}} \tag{28}$$

leads to

$$\epsilon_{\text{heat pump}} = \frac{Q_{ba}}{W_{total}} = \frac{nRT_1 \ln\frac{V_a}{V_b}}{nR(T_2 - T_1)\ln\frac{V_b}{V_a}} = \frac{T_1}{T_1 - T_2} > 1 \tag{29}$$

Since $\epsilon_{\text{heat pump}} > 1$, heat pump works more efficient than converting directly the work given to the heat pump, W_{total}, directly to the thermal energy to heat the room. Note that for heat pumps and refrigerators, ϵ is typically called "Coefficient of Performance", Because these devices move heat rather than convert it into work, their performance is not bounded by 1 like a heat engine's efficiency.