General Physics II: Tutorial Material

Lecture 2 on Chapter 3 and 4 (Gas laws and Kinetic gas theory)

1) When the change of volume, ΔV, with respect to a change of temperature, ΔT, is given by $\Delta V = \beta \Delta T V_0$, where β is the coefficient of volume expansion and V_0 is the initial volume, show that the change in the density is given by $\rho \approx -\beta \Delta T \rho_0$.

Solution:

When the mass of the object is m and the initial volume V_0, the initial density is given by $\rho_0 = m/V_0$. The volume of the object, V, after increase of the temperature ΔT is given by $V = V_0 + \Delta V = V_0(1 + \beta\Delta T)$ and the density

$$\rho = \frac{m}{V} = \frac{m}{V_0(1+\beta\Delta T)} \tag{1}$$

When $\beta\Delta T \ll 1$, the following approximation is valid

$$\frac{1}{1+\beta\Delta T} \approx 1 - \beta\Delta T \tag{2}$$

it follows that

$$\rho = \frac{m}{V_0}(1-\beta\Delta T) = \rho_0 - \beta\Delta T\rho_0 = \rho_0 + \Delta\rho \tag{3}$$

i.e., the change in the density is given by $\Delta\rho = -\beta\Delta T\rho_0$.

Alternative solution:
The first-order Taylor approximation of ρ around V_0 is given by

$$\rho = \rho_0 + \left.\frac{d\rho}{dV}\right|_{V=V_0}(V - V_0) \;\to \Delta\rho = \rho - \rho_0 = \left.\frac{d\rho}{dV}\right|_{V-V_0}\Delta V \tag{4}$$

Therefore:

$$\rho = \frac{m}{V} \;\to \Delta\rho = -\left.\frac{m}{V^2}\right|_{V=V_0}\Delta V = -\frac{m}{V_0^2}\beta\Delta T V_0 = -\beta\Delta T\frac{m}{V_0} = -\beta\Delta T\rho_0 \tag{5}$$

using that $\Delta V = \beta\Delta T V_)$ and $\rho_0 = \frac{m}{V_0}$.

2) Determine formulas for the changes in the surface area and volume of a uniform solid sphere of a radius of r_0 if its coefficient of linear expansion is α and its temperature is changed by ΔT.

Solution:

The surface and the volume of a sphere with a radius r_0 is given by, $S_0 = 4\pi r_0^2$ and $V_0 = 4\pi r_0^3/3$, respectively. When the temperature changes by ΔT, the radius of the sphere changes by $\Delta r = \alpha r_0 \Delta T$, thus the radius is given by $r = (1 + \alpha\Delta T)r_0$. For $1 \gg \alpha\Delta T$, the following approximations are valid:

$$r^2 = (1+\alpha\Delta T)^2 r_0^2 \approx (1 + 2\alpha\Delta T)r_0^2,\; r^3 = (1+\alpha\Delta T)^3 r_0^3 \approx (1+3\alpha\Delta T)r_0^3 \tag{6}$$

leading to the surface and volume to be

$$
\begin{aligned}
S &= 4\pi r^2 \approx 4\pi(1 + 2\alpha\Delta T)r_0^2 = (1 + 2\alpha\Delta T)S_0 \\
V &= 4\pi r^3/3 \approx 4\pi(1 + 3\alpha\Delta T)r_0^3/3 = (1 + 3\alpha\Delta T)V_0
\end{aligned} \tag{7}
$$

Thus, the changes in the surface and volume are given by

$$\Delta S = 2\alpha\Delta T S_0,\ \Delta V = 3\alpha\Delta T V_0, \tag{8}$$

respectively.

Alternative solution:
The same trick as in the alternative solution of Ex. 1 can be used:

$$\Delta S = \frac{dS}{dr}\Big|_{r=r_0} \Delta r = 8\pi r_0 \alpha r_0 \Delta T = 2\alpha\Delta T(4\pi r_0^2) = 2\alpha\Delta T S_0 \tag{9}$$

$$\Delta V = \frac{dV}{dr}\Big|_{r=r_0} \Delta r = 4\pi r_0^2 \alpha r_0 \Delta T = 3\alpha\Delta T(4/3\pi r_0^3) = 3\alpha\Delta T V_0 \tag{10}$$

3) There is an aluminum square plate ($100cm \times 100cm$) at $0°C$ with a hole (in form of a circle) in the center with a radius of 10 cm. If we heat the plate to $500°C$, what will be the size of the plate and how large will be the hole in the center? Note that the coefficient of linear expansion for the aluminum is given by $25\times10^{-6}(°C)^{-1}$, coefficient of volume expansion $75\times10^{-6}(°C)^{-1}$.

Solution:

Using the formula for the linear expansion, $\Delta l = \alpha l_0 \Delta T$, where $l_0 = 100cm$, $\Delta T = (500 - 0)°C = 500°C$ and $\alpha = 25 \times 10^{-6}(°C)^{-1}$, the side of the aluminum plate at $500°C$ is given by,

$$l_{500} = l_0 + \Delta l = 101.25cm \tag{11}$$

Thus, the size of the plate at is given by $101.25cm \times 101.25cm$.

Let us consider an aluminum disk with a radius of 10 cm at $0°C$. This disk closes completely the hole of the aluminum plate at $0°C$. If we warm up the disk to $500°C$, its radius will become,

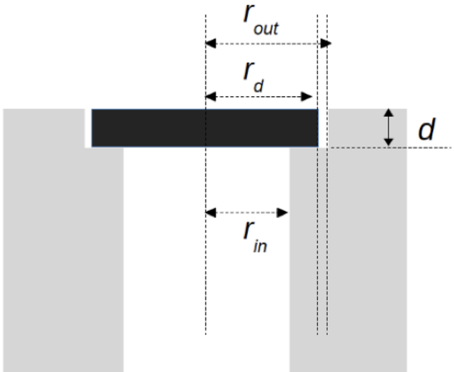
$$r = r_0 + r_0\alpha\Delta T = 10.125cm \tag{12}$$

This disk should fill the hole for the plate at $500°C$. Therefore, the radius of the whole on the plate at $500°C$ is $r = 10.125cm$.

4) A concrete manhole on a road, shown in a photograph below, has an opening that can be closed with a metal disk so that the road surface remains practically even, as illustrated in the figure below showing the side view of the hole. For a given radius, r_d, of the metal disk, r_{in} should be maximized for the comfort of the people who go down the hole and r_{out} must be minimized to reduce the cost. The metal disk has a radius of $r_d = 50cm$ and thickness $d = 2cm$ at $20°C$ and the manhole should be operational, i.e. the disk closes the manhole and the surface practically stays even when placed in the centre, between $-40°C$ and $40°C$.

1. For this problem, we take the linear coefficients for thermal expansion for the metal and concrete to be $10^{-3}/°C$ and $5 \times 10^{-4}/°C$, respectively. Calculate r_{in} and r_{out} when it is constructed at $20°C$.

2. The Young's Moduli for the metal and concrete are $200 \times 10^9 N/m^2$ and $20 \times 10^9 N/m^2$, and the compressive strength $550\times10^6 N/m^2$ and $20\times10^6 N/m^2$, respectively. Temperature in the morning was $35°C$ and the manhole, with the dimension defined above, was closed properly. In the afternoon, the temperature reaches to $45°C$. What will happen to the manhole?



Solution:

1. In general, the formula for linear thermal expansion can be written as

$$r^{T_2} = r^{T_1}(1 + \alpha\Delta T) \tag{13}$$

with $\Delta T = T_2 - T_1$, where r is the radius [m] and T is the temperature [K].
For the disk to be in place for the required temperature range, it is required that at $-40°C$, $r_{in} = r_d$ and at $40°C$, $r_{out} = r_d$.
Using the formula for linear thermal expansion, r_{in} and r_{out} at $20°C$ are given by

$$\begin{aligned} r_{in}^{-40} = r_d^{-40} &\rightarrow\ r_{in}^{20}(1 + \alpha_{concrete}\Delta T) = r_d^{20}(1 + \alpha_{metal}\Delta T),\ \text{with } \Delta T = (-40 - 20)°C \\ r_{out}^{40} = r_d^{40} &\rightarrow\ r_{our}^{20}(1 + \alpha_{concrete}\Delta T) = r_d^{20}(1 + \alpha_{metal}\Delta T),\ \text{with } \Delta T = (40 - 20)°C \end{aligned} \tag{14}$$

From that follows:

$$\begin{aligned} r_{in}^{20} &= r_d^{20}\frac{[1 + (-40 - 20)\alpha_{metal}]}{[1 + (-40 - 20)\alpha_{concrete}]} = 50\frac{1 - 60 \times 10^{-3}}{1 - 60 \times 5 \times 10^{-4}} = 48.454cm \\ r_{out}^{20} &= r_d^{20}\frac{[1 + (40 - 20)\alpha_{metal}]}{[1 + (40 - 20)\alpha_{concrete}]} = 50\frac{1 + 20 \times 10^{-3}}{1 + 20 \times 5 \times 10^{-4}} = 50.495cm \end{aligned} \tag{15}$$

2. When the temperature goes above $40°C$, the metal disk starts to push the concrete hole. By taking into account that the both concrete and metal disk expand, the excess expansion radius between the metal disk and concrete hole is given by

$$\Delta r = r_d^{40}(\alpha_{metal} - \alpha_{concrete})(T - 40)°C \tag{16}$$

For this radius expansion, the metal disk generates thermal stress on the concrete given by

$$\left(\frac{F}{A}\right)_{metal} = E_{metal}\frac{\Delta r}{r_d^{40}} = E_{metal}(\alpha_{metal} - \alpha_{concrete})(T - 40)°C \tag{17}$$

and in the same way, the thermal stress on the disk given by the concrete hole is

$$\left(\frac{F}{A}\right)_{concrete} = E_{concrete}(\alpha_{metal} - \alpha_{concrete})(T - 40)^\circ C \tag{18}$$

where E_{metal} and $E_{concrete}$ are the Young's moduli of the metal and concrete, respectively.

Since we have $E_{metal} > E_{concrete}$, it is always

$$\left(\frac{F}{A}\right)_{metal} > \left(\frac{F}{A}\right)_{concrete} \tag{19}$$

If something breaks, it must be the concrete since the compressive strength of the metal is larger than that of concrete. Therefore, if

$$\left(\frac{F}{A}\right)_{metal} > C_{concrete} \tag{20}$$

the concrete hole breaks, where $C_{concrete}$ is the compressive strength of the concrete (=maximum compressive pressure on concrete before breaking). With the given values, we obtain

$$\left(\frac{F}{A}\right)_{metal} = 5 \times 5 \times 10^{-4} \times 200 \times 10^9 N/m^2 = 5 \times 10^8 N/m^2 \tag{21}$$

thus

$$\left(\frac{F}{A}\right)_{metal} > C_{concrete} = 20 \times 10^6 N/m^2 \tag{22}$$

and the concrete hole breaks and metal disk remains unbroken at $45^\circ C$. For completeness, we give

$$\left(\frac{F}{A}\right)_{concrete} = 5 \times 5 \times 10^{-4} \times 20 \times 10^9 N/m^2 = 5 \times 10^7 N/m^2 \tag{23}$$

which is one order of magnitude less than $(F/A)_{metal}$.

5) Calculate the density of nitrogen at STP using the ideal gas law. Note that the nitrogen atom has Z= 7 and A = 14 and the nitrogen gas molecule is N_2.

Solution:

By considering the nitrogen gas to be an ideal gas, the volume of the n mole nitrogen gas is given by

$$V = \frac{nRT}{P} \tag{24}$$

At STP, i.e. $T = 0^\circ C = 273K$ and $P = 1atm$, this leads to

$$V = n \times 22.4 \times 10^{-3} m^3 \tag{25}$$

Since the atomic number of the nitrogen is 14 and the nitrogen gas molecule is made of two nitrogen atoms, the mass of n mole nitrogen gas is given by $m = n \times 28g$. Therefore, the density of the gas is given by

$$\rho = \frac{m}{V} = \frac{28g}{22.4 \times 10^{-3} m^3} = 1.25 \times 10^{-6} g/mm^{-3} \tag{26}$$

6) A storage tank contains 21.6 kg of N_2 gas at an absolute pressure of 3.85 atm. What will be the pressure if the nitrogen is replaced by an equal mass of CO_2 at the same temperature?

Solution:

We consider N_2 and CO_2 gases to be an ideal gas. The temperature of the nitrogen gas, T_N, in the container with a volume V_0 is given by

$$T_N = \frac{P_N V_0}{n_N R} \tag{27}$$

where P_N and n_N are the pressure and the mole number of the N_2 gas. If the gas of the container is replaced by the CO_2 with the same temperature and equal mass, the pressure of the CO_2 is given by

$$P_C = \frac{n_C R}{V_0} T_C = \frac{n_C R}{V_0} T_N = \frac{n_C R P_N V_0}{V_0 n_N R} = \frac{n_C}{n_N} P_N \tag{28}$$

where the subscript "C" denotes for "CO_2". The ratio of the mole numbers of the two gasses is given by

$$\frac{n_C}{n_N} = \frac{m_C}{mm_C} \frac{mm_N}{m_N} \tag{29}$$

where mm_C and mm_N are the molecular mass of CO_2 and N_2, respectively, and m_C and m_N are the masses for CO_2 and N_2 which are identical and 21.6 kg. The atomic mass of N_2 is 28 and CO_2, $12 + 2 \times 16 = 44$. Therefore

$$\frac{n_C}{n_N} = \frac{28}{44} \tag{30}$$

The pressure for CO_2 is now given by

$$P_C = \frac{n_C}{n_N} P_N = \frac{28}{44} \times 3.85 atm = 2.45 atm \tag{31}$$

7) A space ship enters in the earth atmosphere with a speed of 10km/second. Atmosphere molecules (assume nitrogen) then strike the nose of the space ship with this speed. What is the corresponding temperature? Note that the mass of one nitrogen atom is $2.3 \times 10^{-26} kg$.

Solution:

In the rest frame of the space ship, we can consider that the gas molecules are moving with an rms-velocity of 10 km/second. From the kinetic theory of gasses, we have

$$P \cdot V = \frac{N \cdot m \cdot v_{rms}^2}{3} \tag{32}$$

where m is the mass of the gas molecule, v_{rms} is the rms-speed of the gas molecule, and N is the number of the gas molecules. Assuming that the nitrogen gas is an ideal gas, the ideal gas law gives

$$P \cdot V = N \cdot k \cdot T \tag{33}$$

where T is the temperature of the gas and k is the Boltzmann's constant. A combination of the two leads to

$$\frac{m \cdot v_{rms}^2}{3} = k \cdot T \tag{34}$$

The mass of the nitrogen gas molecule, N_2, is given by

$$m = 2 \times (2.3 \times 10^{-26} kg) = 4.6 \times 10^{-26} kg \tag{35}$$

and the Boltzmann's constant,

$$k = \frac{R}{N_A} = \frac{8.4 J/mol\ K}{6 \times 10^{23}/mol} = 1.4 \times 10^{-23} \frac{J}{K} \tag{36}$$

Therefore, the temperature of the N_2 gas is given by

$$T = \frac{m \cdot v_{rms}^2}{3k} = \frac{4.6 \times 10^{-26} kg (10 \times 10^3 m/sec)^2}{3 \times 1.4 \times 10^{-23} J/K} = \frac{4.6 \times 10^{-18} J}{4.2 \times 10^{-23} J/K} \approx 10^5 K. \tag{37}$$